COUPLED WAVES AND FLOQUET APPROACH TO PERIODIC STRUCTURES

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ABSTRACT

This report compares the theoretical and numerical results of Floquet theory and coupled mode theory at first order Bragg interactions. It also investigates second order interactions where only Floquet theory holds. Coupled mode theory is shown to give close agreement with the exact solution even as $n \to 1$. The Floquet theory predicts a shift in the bandgap away from the Bragg condition which is not predicted by the coupled mode theory. The limitations and accuracy of the numerical solution of the exact theory is discussed.

CHAPTER I

INTRODUCTION

Wave propagation in periodic structures has applications to many areas in engineering and physics. Recently much interest in electromagnetic wave propagation in periodic structures has centered on integrated optical devices such as filters, couplers and distributed feedback (DFB) lasers. For this reason, we consider in this report the example of transverse electromagnetic (TEM) waves propagating in an unbounded media with spatially varying dielectric constant.

To gain physical insight into the effect of the periodic structure upon incident waves we consider a TEM wave traveling in a periodic media as illustrated in Fig. 1. The medium is periodic at distance Λ and the incident plane wave has wavelength λ. The scattered rays will combine to produce maximum intensity if the path difference from adjacent perturbations are an integral number of wavelengths, or if

$$2\Lambda \sin\theta = n\lambda \qquad (n = 1,2,3,\ldots) \qquad (1.1)$$

where the dielectric constant of the media is assumed to be unity. Equation (1.1) is called Bragg's Law and gives the conditions for the spacing of the periodic perturbations of the medium to provide strong scattering. However, it does not give the amplitude of the scattered plane wave.

Two theories in particular have been used to calculate wave amplitudes. The first, coupled mode theory, provides a relatively simple method for calculating the scattered wave amplitude and will be shown to be an excellent approximation in the first Bragg condition ($n = 1$ in Eq. (1.1)) if the periodic disturbance is physically reasonable. Unfortunately, coupled mode theory does not provide us with estimates

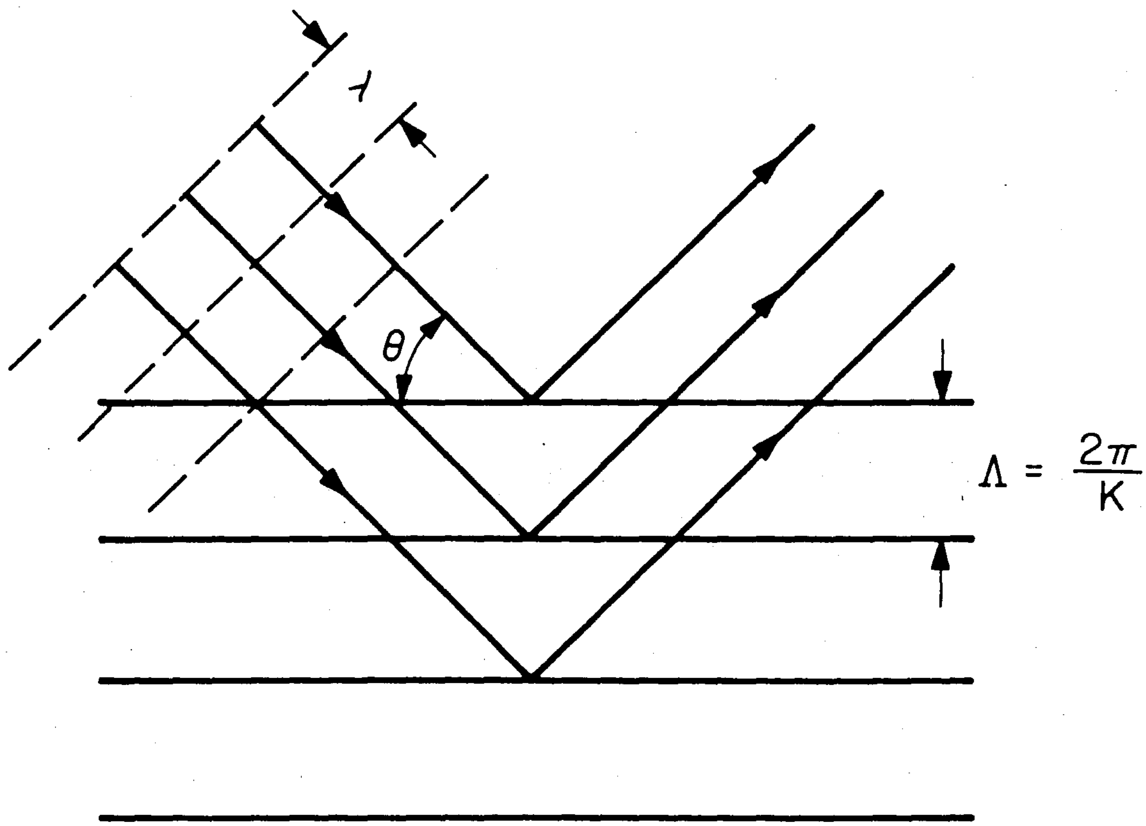

Fig. 1. Bragg scattering of electromagnetic wave from periodic media.

of the scattering amplitude for higher order Bragg scattering $n > 1$ (in Eq. (1.1)). A second theory, using the Floquet form of solution, gives exact results for all orders of Bragg scattering but is more complicated and perhaps offers less physical insight into the problem than does the coupled mode theory.

The purpose of this report is two-fold:

1) To numerically compare the exact Floquet theory with the approximate coupled mode theory so that the limitations of the latter and the new phenomena of the former may be observed.

2) To investigate higher order regions of Bragg scattering using the exact Floquet analysis from which coupling coefficients can be derived and used in a modified coupled mode analysis.

This second purpose has had increased importance recently since it is not always technologically possible to make a structure periodic at the first Bragg condition. Extensions of the use of optical devices into the ultraviolet regions should be possible using higher order Bragg coupling and has already been experimentally demonstrated by Bjorkholm and Shank[2]. At X-ray frequencies use of the natural periodicities of crystal structures has been proposed to provide DFB at the first Bragg condition[3].

This report is self-contained and therefore starts from first principles. Chapter II contains the dispersion relation for waves in a periodic media as derived from the Floquet analysis which is then cast into a simple form for numerical analysis. Coupled mode theory is also derived from the exact analysis. Chapter III contains the numerical results of the preceding theories with a discussion of the resulting phenomena.

CHAPTER II

A. The Exact Floquet Solution

The basic problem to be considered is that of a TEM plane wave incident upon a slab with a periodic dielectric constant. We suppose that the slab is transversely unbounded but the analysis and qualitative results may be altered to account for guiding structures[5].

Figure 2 shows the case under consideration. Assuming a time dependence of $e^{-i\omega t}$ we have from Maxwell's equations,

$$\nabla \times \underline{E}(z,t) = i\mu_o\omega \, \underline{H}(z,t) \tag{2.1}$$

$$\nabla \times \underline{H}(z,t) = -i\varepsilon'\omega \, \underline{E}(z,t) \tag{2.2}$$

$$\frac{\partial^2 E(z,t)}{\partial z^2} + k^2\varepsilon[1 + \eta f(Kz)]E(z,t) = 0 \tag{2.3}$$

where ω = radian frequency

μ_o = free space permittivity

$\varepsilon' = \varepsilon_o\varepsilon$ = dielectric constant

ε_o = free space dielectric constant

ε = relative dielectric constant

$k = \omega/c$ = free space wavenumber

and where $f(Kz) = \sum_{n=0}^{\infty} f_n \cos(nKz)$ expresses the periodicity of the dielectric constant with f_n real and $f_o \equiv 0$. Both the relative dielectric constant ε' and the perturbation η may have positive or negative imaginary parts. E represents one of the transverse components of the electric field.

The Floquet form of the solution to (2.3) is

$$E(z,t) = E(z)e^{-i\omega t} \tag{2.4}$$

$$E(z) = \sum_{n=-\infty}^{\infty} a_n e^{i(\beta+nK)z} \tag{2.5}$$

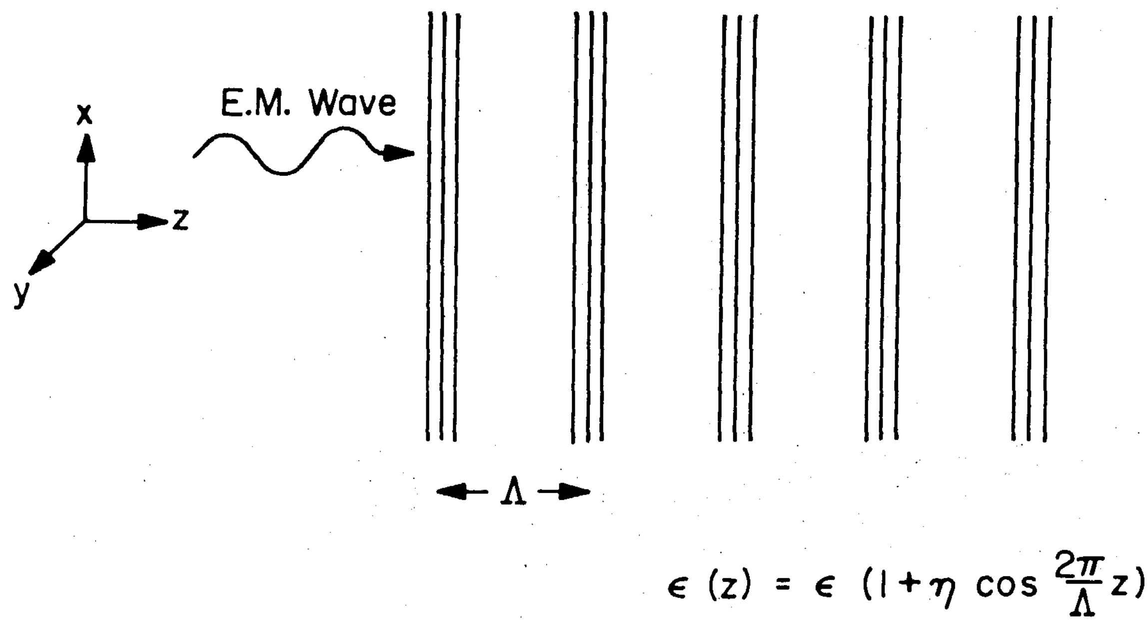

Fig. 2. Electromagnetic wave propagating in infinite, unbounded periodic media.

This form represents the electric field as a field with periodicity equal to that of the medium. Each term $a_n\, e^{i[(\beta+nK)z - \omega t]}$ is called a space harmonic of order n and amplitude a_n. The space harmonics may be either forward or backward traveling waves, depending on the value of n and β.

Substituting (2.5) into (2.3), we find

$$\sum_{n=-\infty}^{\infty}(-a_n)(\beta+nK)^2 e^{i(\beta+nK)z} + \epsilon k^2 \sum_{n=-\infty}^{\infty} a_n e^{i(\beta+nK)z} + \epsilon k^2 \sum_{n=-\infty}^{\infty} a_n \left(\frac{\eta}{2}\right) e^{i(\beta+nk)z} \sum_{m=-\infty}^{\infty} f_m e^{imKz} = 0 \qquad (2.6)$$

where $f_m = f_{-m}$. Upon rearranging we have for each n,

$$\{[\epsilon k^2 - (\beta+nK)^2]a_n + \frac{\eta}{2}\epsilon k^2 \sum_{m=-\infty}^{\infty} f_m a_{n-m}\}\, e^{i(\beta+nK)z} = 0 \qquad (n = 0,\pm 1,\pm 2,\ldots) \qquad (2.7)$$

This can be expressed as

$$D_n a_n + \sum_{m=-\infty}^{\infty} a_{n-m} f_m = 0 \qquad (n = 0,\pm 1, \pm 2,\ldots) \qquad (2.8)$$

where

$$D_n \equiv \frac{2}{\eta}\left[1 - \frac{(\beta+nK)^2}{\epsilon k^2}\right] \qquad (n = 0,\pm 1,\pm 2,\ldots) \qquad (2.9)$$

It should be noted that for guiding media or propagation in periodic plasmas, the form of D_n is slightly changed.

Equation (2.8) represents an infinite number of homogeneous linear algebraic equations. It is well known that the nontrivial solution requires the determinant of the coefficients of a_n to vanish.

Before solving this equation, it is useful to look at the form of the matrix involved for a simple periodicity. Suppose that the dielectric perturbation is cosinusoidal. In this case there are only two nonzero coefficients $f_{\pm 1} = 1$. Equation (2.8) becomes

$$D_n a_n + a_{n+1} + a_{n-1} = 0 \qquad (n = 0, \pm 1, \pm 2, \ldots) \tag{2.10}$$

or

$$\underline{\underline{D}} \cdot \underline{a} = 0 \tag{2.11}$$

where

$$\underline{\underline{D}} \equiv \begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & D_{-2} & 1 & & & \\ & 1 & D_{-1} & 1 & & \\ & & 1 & D_0 & 1 & \\ & & & 1 & D_1 & 1 & \\ & & & & 1 & D_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}$$

$$\underline{a} \equiv \begin{pmatrix} \vdots \\ a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix}$$

and the nontrivial solution is

$$\det \underline{\underline{D}} = 0 \tag{2.12}$$

The dispersion relation (2.12) relates β and k. One can then solve for the relative space harmonic amplitudes a_n from (2.10). For problems involving distributed feedback, (2.12) must be solved first and then the equation for the reflection coefficient can be solved from the boundary condition.

Equation (2.12) can be solved by several different methods including the use of continued fractions[8] and Hill's equation[6,7,9]. The latter method is used in this report since our primary interest is in the explicit relation of β and k which the Hill's equation method provides in terms of an appropriately truncated infinite determinant. Appendix A shows how the dispersion relation (2.12) can be recast into the form

$$\sin^2\left(\frac{\pi\beta}{K}\right) = \Delta(0)\sin^2\left(\frac{\pi k}{K}\sqrt{\varepsilon}\right) \tag{2.13}$$

where

$$\Delta(0)_{mn} = \begin{cases} 1 & m=n \\ \dfrac{-k^2\varepsilon}{m^2K^2-k^2\varepsilon}\ \dfrac{\eta}{2}\ f_{|n-m|} & m\neq n \end{cases}$$

$$\Delta(0) = \det \underline{\underline{\Delta}}(0)$$

If the matrix $\underline{\underline{\Delta}}(0)$ can be truncated to p^{th} order where p is small, then (2.13) is easily solved explicitly for β in terms of k. The appropriate branches which result from the square root and arcsine functions are chosen to correspond to the proper unperturbed solutions. We find that near the first order Bragg interaction (i.e. n=1 in (1.1)) equation (2.12) does not present numerical problems for $\eta \gtrsim 0.01$. This is discussed further in Chapter III.

B. The Coupled Mode Solution

In Section A we saw that the Floquet form requires us to solve Eq. (2.12) or some equivalent operation to obtain the dispersion relation. In this section we find a less complex but approximate dispersion relation.

If we take η finite but small, we can write the electric field expression approximately as[5]

$$E_y = (a_o + a_1e^{iKz} + a_{-1}e^{-iKz})e^{i\beta z} \tag{2.14}$$

where we neglect all other space harmonics. The system of equations (2.14) becomes

$$D'_o a_o + \frac{\eta}{2} a_1 + \frac{\eta}{2} a_{-1} = 0 \tag{2.15a}$$

$$D'_1 a_1 + \frac{\eta}{2} a_o = 0 \tag{2.15b}$$

$$D'_{-1} a_{-1} + \frac{\eta}{2} a_o = 0 \tag{2.15c}$$

where

$$D'_n = \frac{\eta}{2} D_n = 1 - \left(\frac{\beta+nK}{k\sqrt{\varepsilon_1}}\right)^2 \tag{2.16}$$

For most values of k, D'_{-1} and D'_1 are large compared to η, and therefore from (2.15 b,c) we see that a_1 and a_{-1} are small compared with a_o and to satisfy (2.15a) we must have

$$D'_o \simeq 0 \tag{2.17}$$

which gives the same dispersion equation as in the unperturbed medium:

$$\beta = \sqrt{\varepsilon_1}\, k \tag{2.18}$$

However, it is possible that not only $D'_o \simeq 0$, but simultaneously D'_1 or $D'_{-1} \simeq 0$. Let us suppose the parameters of the problem are such that

$$D'_o \simeq 0 \longrightarrow \beta^2 = k^2 \varepsilon_1 \tag{2.19}$$

and

$$D'_{-1} \simeq 0 \longrightarrow (\beta - K)^2 = k^2 \varepsilon_1 \tag{2.20}$$

From (2.19) and (2.20) this occurs if

$$\beta^2 - 2\beta K + K^2 \simeq \beta^2 \tag{2.21}$$

which requires

$$\beta = K/2 \tag{2.22}$$

In this case we still have $a_1 << a_o$, but we can no longer say $a_{-1} << a_o$. The system of equations for the region around the interaction point ($k_o = \omega_o/c$, $_o = \sqrt{\epsilon_1}\, k_o$, $\beta_o = K/2$) becomes

$$D'_o a_o + \frac{\eta}{2} a_{-1} = 0 \qquad (2.23a)$$

$$D'_{-1} a_{-1} + \frac{\eta}{2} a_o = 0 \qquad (2.23b)$$

The nontrivial solution of (2.23a,b) requires

$$D'_o D'_{-1} = \frac{\eta^2}{4} \qquad (2.24)$$

In the neighborhood of the intersection point we can write

$$\beta = \beta_o + \Delta\beta \qquad (2.25a)$$

$$k = k_o + \Delta k \qquad (2.25b)$$

to give

$$D'_o D'_{-1} = [1 - (\frac{\beta_o + \Delta\beta}{(k_o + k)\sqrt{\epsilon_1}})^2][1 - (\frac{-\beta_o + \Delta\beta}{(k_o + \Delta k)\sqrt{\epsilon_1}})^2] = \frac{\eta^2}{4} \qquad (2.26)$$

Expanding and using

$$(1 + \xi_1)\Big/(1 + \xi_2) = 1 + \xi_1 - \xi_2 \qquad (2.27)$$

for ξ_1, ξ_2 small, we obtain

$$(\frac{\Delta\omega}{\omega_o})^2 - (\frac{\Delta\beta}{\beta_o})^2 = (\frac{\eta}{4})^2 \qquad (2.28)$$

For $|\Delta\omega/\omega_o| < \eta/4$, we write equation (2.30) as

$$\frac{\Delta\beta}{\beta_o} = \pm i\sqrt{(\frac{\eta}{4})^2 - (\frac{\Delta\omega}{\omega_o})^2} = \pm i\sqrt{(\frac{\eta}{4})^2 - (\frac{\Delta k}{k_o})^2} \qquad (2.29)$$

which is the equation of an ellipse. This region is called a stopband interaction and it corresponds to an exponential energy transfer between the two interacting waves as $\Delta\beta$ is imaginary. The coupling coefficient χ is defined as the maximum of the imaginary part of β, for this case:

$$\chi = \Delta\beta\Big|_{\Delta\omega=0} = \frac{\sqrt{\varepsilon}\, k\eta}{4} \tag{2.30}$$

For $\Delta\omega/\omega_o| > \eta/4$, Eq. (2.28) may be writte

$$\frac{\Delta\beta}{\beta_o} = \pm\sqrt{\left(\frac{\Delta\omega}{\omega_o}\right)^2 - \left(\frac{\eta}{4}\right)^2} = \sqrt{\left(\frac{\Delta k}{k_o}\right)^2 - \left(\frac{\eta}{4}\right)^2} \tag{2.31}$$

which corresponds to a hyperbola with asymptotes corresponding to the unperturbed ($\eta = 0$) medium. The Brillouin diagram near an intersection region is shown in Fig. 3.

Using equation (2.23a), we can calculate the ratios of the amplitudes of the space harmonics:

$$\left|\frac{a_{-1}}{a_o}\right| = \left|\frac{2}{\eta} D'_o\right| = \left|\frac{4}{\eta}\left[\frac{\Delta\beta}{\beta_o} - \frac{\Delta\omega}{\omega_o}\right]\right| \tag{2.32}$$

Thus, by assuming a simple solution (2.14) to the wave equation we have found the coupled mode dispersion relation (2.29) which should approximate the previous exact dispersion relation (2.12) or (2.13).

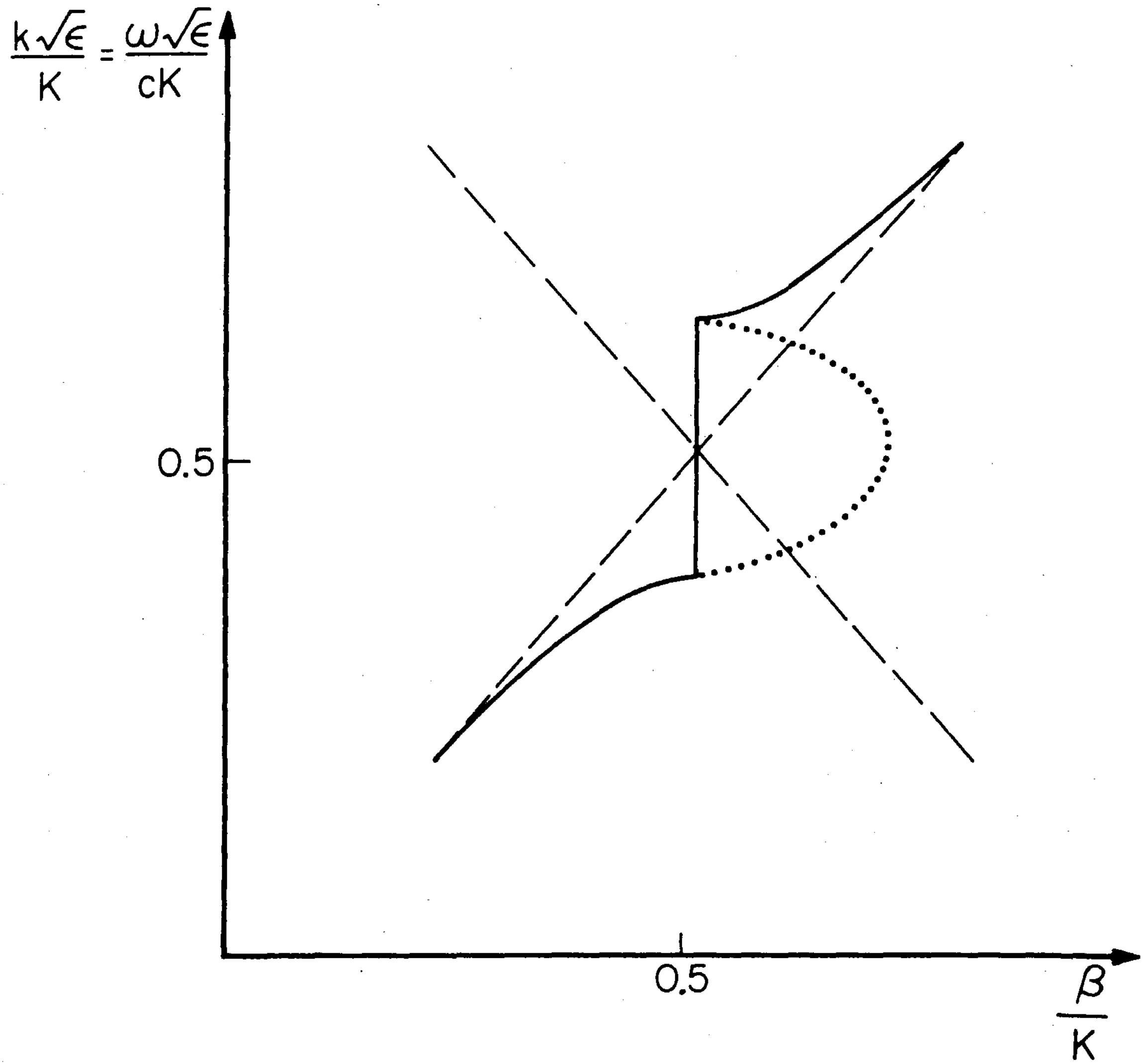

Fig. 3. Brillouin diagram near first Bragg interaction. Dotted line is imaginary part of β/K. Dashed lines are for unperturbed media where $\eta = 0$.

CHAPTER III

DISCUSSION OF BRAGG INTERACTIONS

A. Discussion

The Brillouin diagram is a useful tool for investigating wave propagation in periodic media. In general this diagram shows the variation of frequency with wavenumber, that is, the Brillouin diagram is a graph of the dispersion relation. For periodic media we find both regions of freely propagating waves (passbands) and regions of attenuated propagating waves (stopbands or bandgaps). Additionally, we find the magnitude of the coupling coefficients used in the coupled mode theory. This coefficient is proportional to the maximum imaginary part of the wavenumber in the bandgap. We can also compare the results of Floquet theory with the analytic results for the bandgap region derived from the coupled mode theory. Casey et. al.[4] point out that for the case of dispersive media, the matrix in the dispersion equation (2.13) need never be larger than 15 x 15 for precision of three significant figures for β near the first Bragg interaction. However, Casey et. al.[4] considered only the cases η = 1.0, 0.1. It can be shown by examination of (2.13) that in the bandgap region very slight changes in the value of $\Delta(0)$ may drastically change the character of the bandgap for $\eta \to 0.001$. Thus, the truncation of $\underline{\Delta}(0)$ presents additional problems unless $\eta \gtrsim 0.01$. For $\eta \lesssim 0.01$ one should use other numerical methods or approximate theories. In our case we used 19 x 19 size matrices which could have been decreased considerably for $0.01 \lesssim \eta \lesssim 0.1$. The diagrams (Figs. 4-8) are all plots of normalized frequency ($\omega\sqrt{\varepsilon}/ck = k\sqrt{\varepsilon}/K$; $K = 2\pi/\Lambda$) versus normalized wavenumber (β/K) with separate scales for the imaginary part of the wavenumber. In most of the cases the plots are expanded views around Bragg interaction regions of order n where $\beta/K \sim k\sqrt{\varepsilon}/K \sim n/2$.

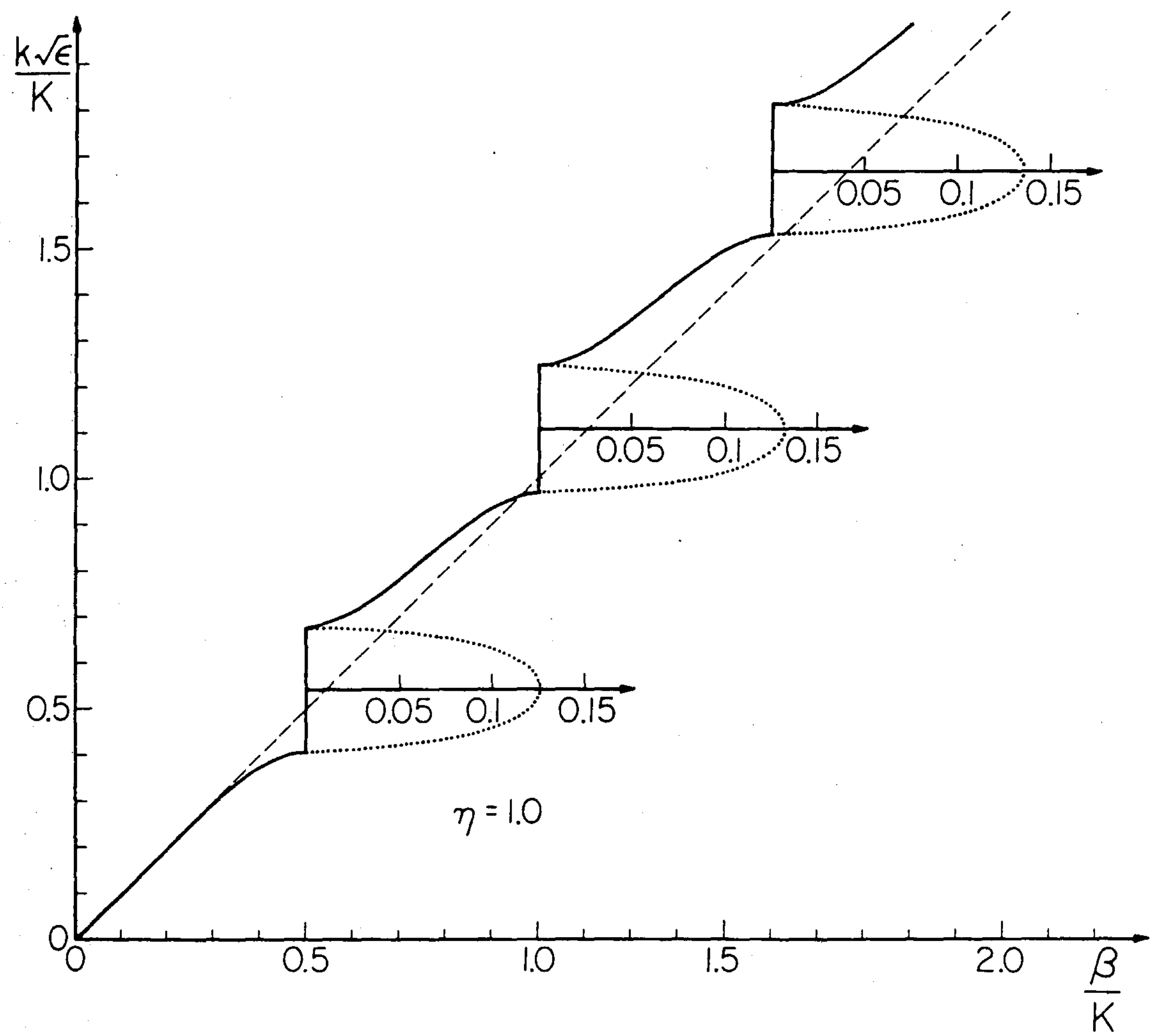

Fig. 4. Brillouin diagram of first three Bragg interactions with $\eta = 1$. Dotted line is imaginary part of β/K.

Thus, the second order Bragg interaction is at $\beta_0/K = k_0\sqrt{\varepsilon}/K = 1.0$ with $\Delta\beta/K$ and $\Delta k\sqrt{\varepsilon}/K$ representing the amount of wavenumber or frequency deviation from the exact Bragg condition.

Fig. 4 demonstrates the main features of Floquet theory for the extreme case of $\eta = 1$. As expected from the physical explanation of Chapter I the largest effect of the periodicity is in the vicinity of the Bragg frequencies $\omega\sqrt{\varepsilon}/cK = n/2$. However, each bandgap is shifted toward a higher frequency and this shift becomes more pronounced with increasing order. This gradual shifting of the Brillouin diagram above the line $k\sqrt{\varepsilon}/K = \beta/K$ indicates that the periodic medium slightly decreases the phase velocity away from the bandgap. In other words, the periodic medium slightly increases the effective dielectric constant of the medium. The slope or group velocity changes slightly from the value of that of the unperturbed medium at the center of the passbands and smoothly goes to zero at the passband edges. In the bandgap there is both a real and an imaginary part of the wavenumber. The fact that β is complex indicates waves that propagate with decreasing amplitude due to coupling.

Fig. 5 is the expansion of the first order Bragg interaction region of Fig. 4. Superimposed is the approximate coupled mode relation from equation (2.29). Note that even for $\eta = 1$, the coupled mode theory closely predicts the correct coupling coefficient. However, the bandgap shift shown by the Floquet theory is not predicted by coupled mode theory for first order Bragg interactions. The coupled mode result (lower curve of Fig. 5) can be used for η other than $\eta = 1$ by multiplying each scale by η.

Fig. 6 shows the first two Bragg interactions for $\eta = 0.1$. It appears that for low order Bragg interactions the coupling is proportional

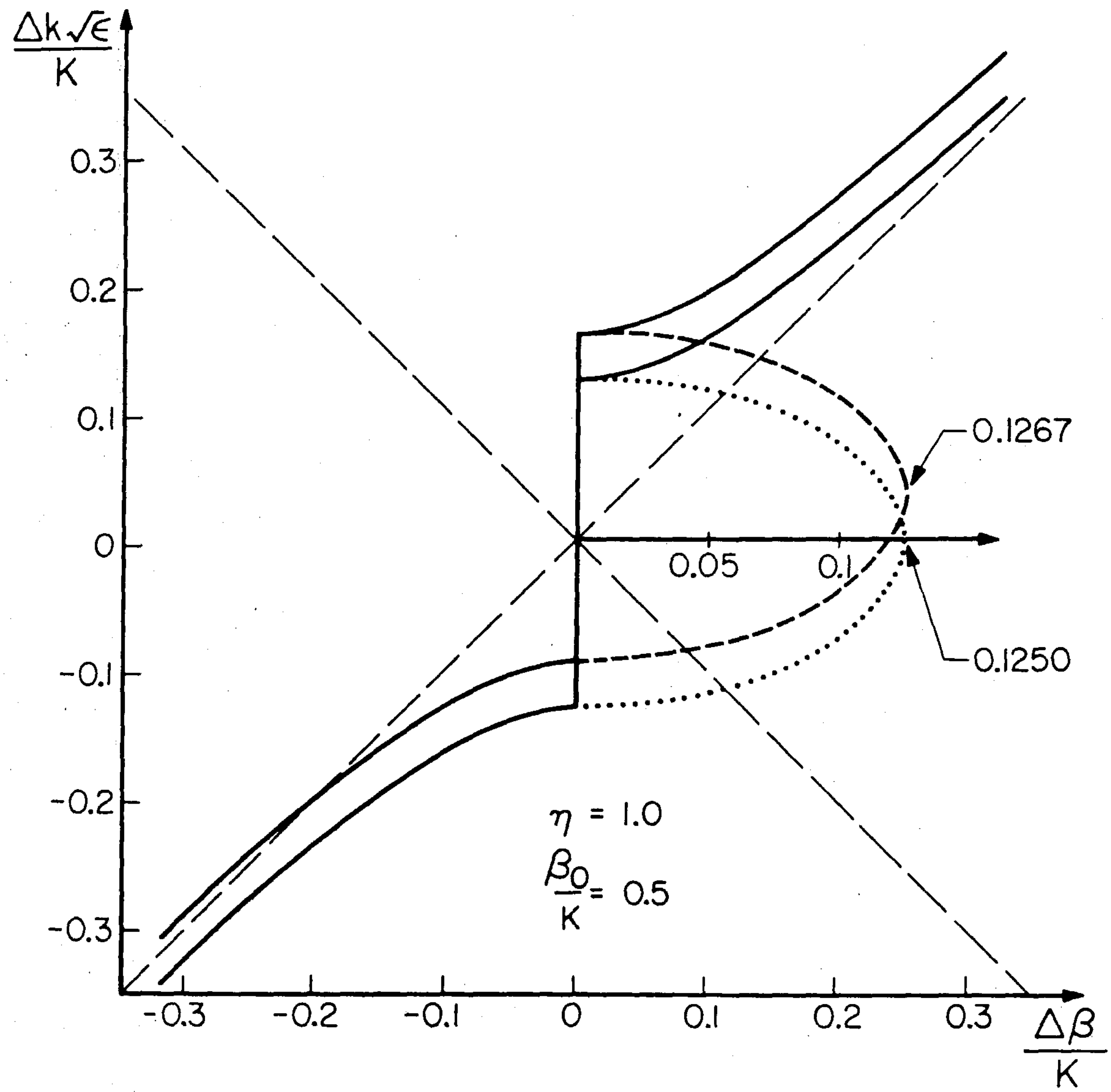

Fig. 5. Brillouin diagram of first Bragg interaction with $\eta = 1$. This compares Floquet theory (upper dashed curve) with coupled mode theory (lower dotted curve). Dotted and heavily dashed lines are imaginary parts of β/K.

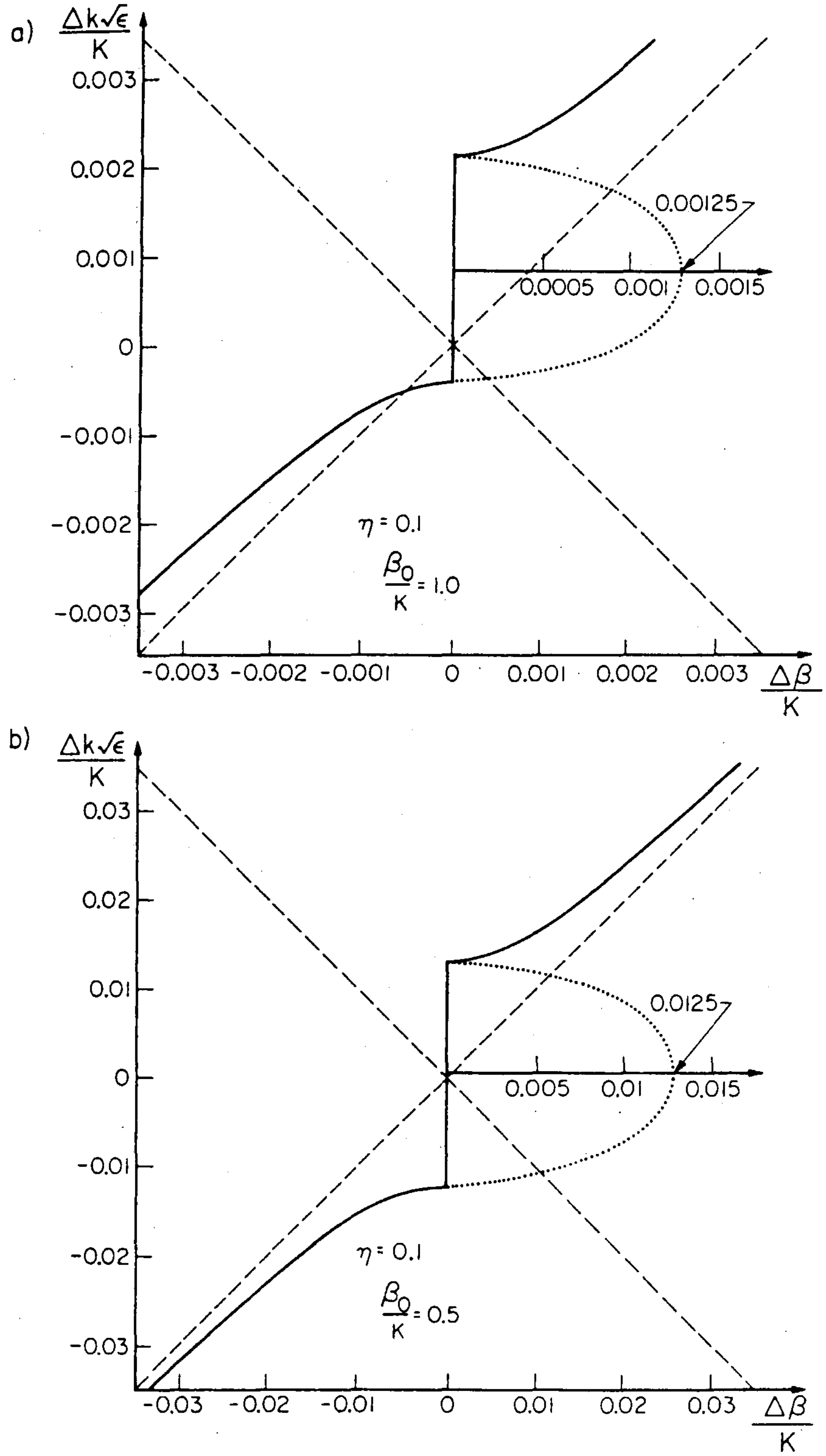

Fig. 6. Brillouin diagram of a) second and b) first of Bragg interactions for η = 0.1. Dotted lines are imaginary parts of β/K.

to η^n where η is the maximum perturbation and n is the Bragg order and that the bandgap shift decreases with decreasing η. The maximum imaginary value of β shown for n = 1 is less than 0.1% larger than that predicted by the simple coupled mode theory.

Fig. 7 shows a comparison of plots at n = 1 for η = 1.0, 0.1, 0.01. The scales are decreased by a factor of ten for each of the respective cases. The most striking result is the close match between the full Floquet theory and the coupled mode theory especially for $\eta \lesssim 0.1$ in predicting coupling. For $\eta \lesssim 0.01$ the two theories also agree on the bandgap shift. This latter case covers many practical examples in guided wave optics and DFB lasers but approaches the region of η where previously described numerical problems may occur in equations (2.13).

Fig. 8 shows the relative contributions of the space harmonics. The upper and lower curves use a 3 x 3 and 5 x 5 matrix respectively in the dispersion relation. The upper curve thus represents the coupling of the a_{-1}, a_0 and a_{+1} space harmonics whereas the lower curve also couples the a_{-2} and a_{+2} space harmonics. The differences are small and the 5 x 5 matrix gives dispersion characteristics that are $\lesssim$ 1% different from that of the 19 x 19 matrices.

B. Conclusions

This report has compared the theoretical and numerical results of Floquet theory and coupled mode theory at first order Bragg interactions and investigated second order interactions where only Floquet theory holds. We found that coupled mode theory gives surprisingly good approximations for the coupling coefficients even as $\eta \to 1$ although only the Floquet theory shows bandgap shifts away from the exact Bragg conditions

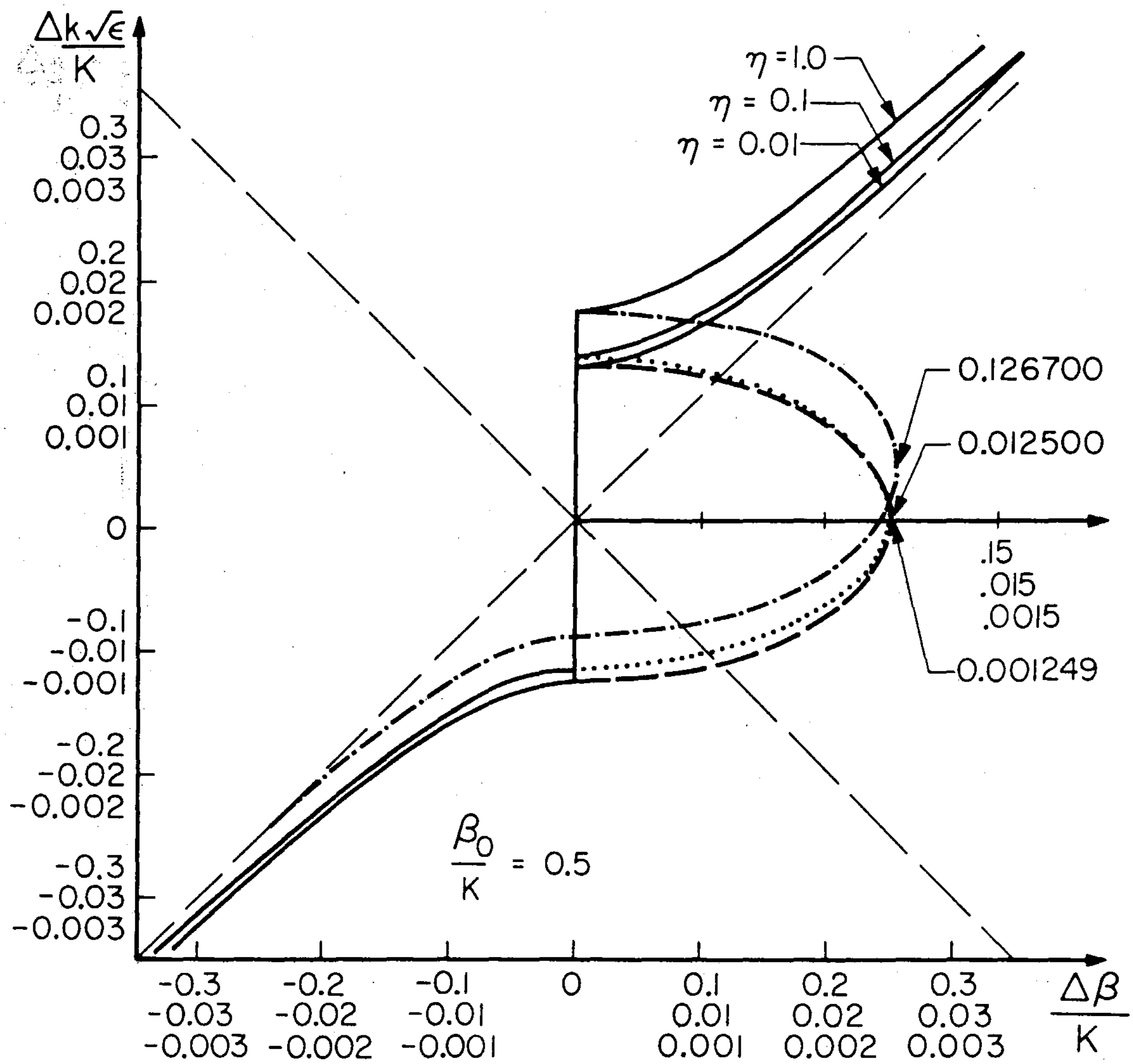

Fig. 7. Brillouin diagram of first Bragg interaction for $\eta = 1$ (top curve), $\eta = 0.1$ (middle curve) and $\eta = 0.01$ (bottom curve). Note difference in scales for each case. Imaginary β/K are elliptical regions with separate scale.

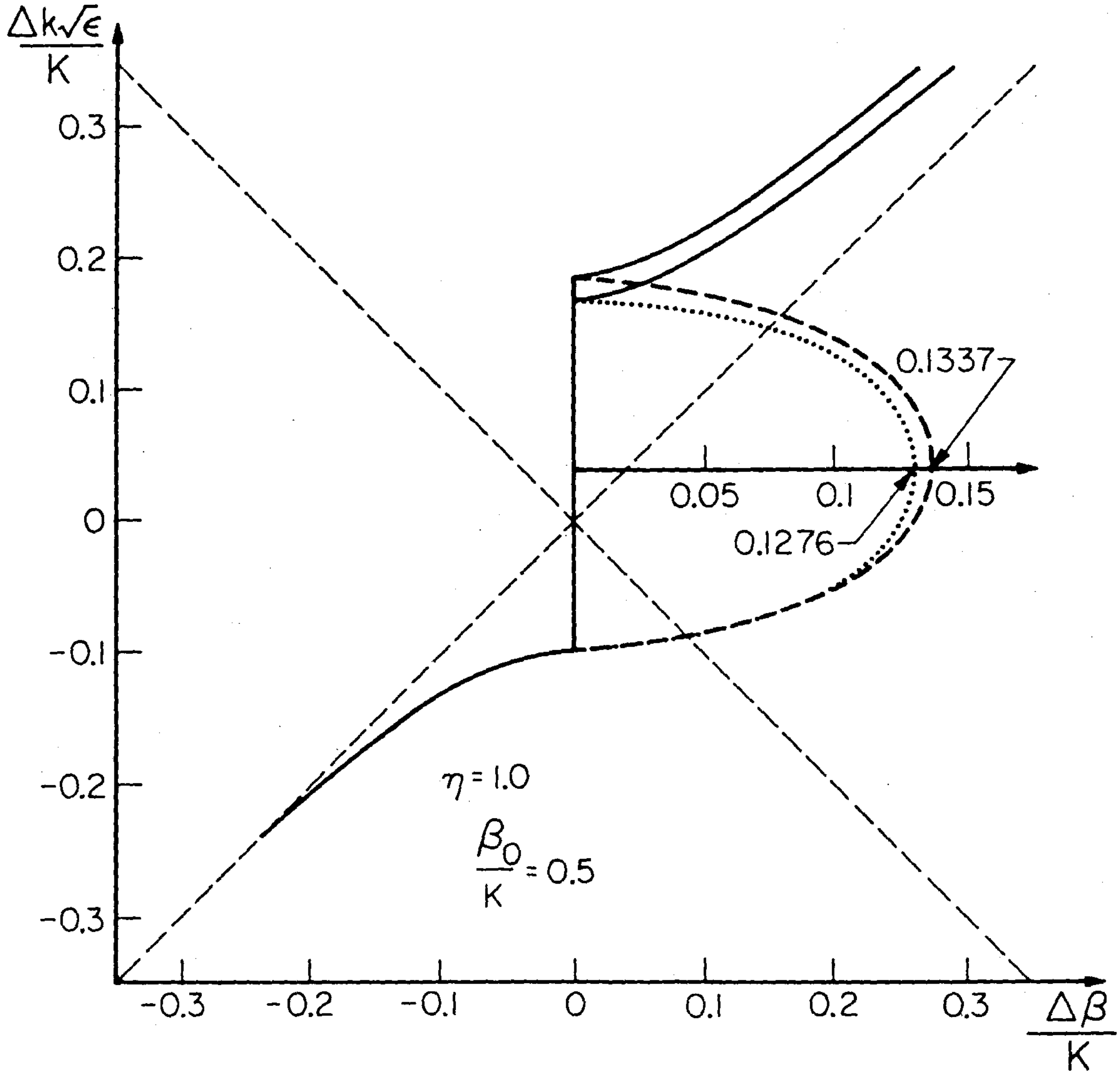

Fig. 8. Brillouin diagram of first Bragg interaction for $\eta = 1$ for 3 x 3 (upper dashed line) and 5 x 5 (lower dotted line) matrix.

and the resulting decrease in phase velocity. Furthermore, we found that relatively small 5 x 5 matrices were sufficient in the truncated Floquet theory for 1% accuracy if $\eta \gtrsim 0.01$ when we used the Hill's determinant. It was noted that for $\eta \lesssim 0.01$ one should use coupled mode theory (for first order Bragg) or perhaps continued fractions to avoid large computer times and numerical complications. Finally we note that distributed feedback systems designed for higher order feedback may be analyzed using the truncated Floquet solution in the bandgap as the coupling coefficient in standard coupled mode analysis.

APPENDIX A

HILL'S EQUATION DERIVATION

Here we sketch the steps leading to Hill's determinant (2.13) by starting with (2.8,9). We follow the method of Whittaker and Watson[6].

$$D_n a_n + \sum_{m=-\infty}^{\infty} a_{n-m} f_m = 0 \qquad (n = 0, \pm 1, \pm 2, \ldots) \qquad \binom{2.8}{A.1}$$

where

$$D_n \equiv \frac{2}{\eta}\left[1 - \frac{(\beta + nK)^2}{\varepsilon\, k^2}\right] \qquad (n = 0, \pm 1, \pm 2, \ldots) \qquad \binom{2.9}{A.2}$$

or equivalently,

$$\det \underline{\underline{D}}' = 0 \qquad (A.3)$$

where

$$\underline{\underline{D}}' = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \ldots f_{-1} & D_{-1} & f_1 & f_2 & f_3 \ldots \\ \ldots f_{-2} & f_{-1} & D_o & f_1 & f_2 \ldots \\ \ldots f_{-3} & f_{-2} & f_{-1} & D_1 & f_1 \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

To secure convergence of an infinite determinant it is required that the product of the diagonal elements converge absolutely and that the sum of the non-diagonal elements converge absolutely. To satisfy these conditions, we can divide each m^{th} row of $\underline{\underline{D}}'$ by a function of m without changing the relation between β and k in the dispersion equation. To do this we define two matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$ with elements A_{mn} and B_{mn} respectively.

$$A_{mn}(\beta) \equiv \begin{cases} \dfrac{(\beta + mk)^2 - k^2\varepsilon}{m^2K^2 - k^2\varepsilon} & m = n \\[2ex] \dfrac{-k^2\varepsilon}{m^2K^2 - k^2\varepsilon}\,\dfrac{\eta}{2}\, f_{|m-n|} & m \neq n \end{cases} \qquad (A.4)$$

$$\beta_{mn}(\beta) \equiv \begin{cases} 1 & m = n \\ \dfrac{-k^2\varepsilon}{(\beta+mK)^2 - k^2\varepsilon} \; \dfrac{\eta}{2} \; f_{|m-n|} & m \neq n \end{cases} \qquad \text{(A.4 cont'd.)}$$

The determinants of $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are respectively $\Delta(\beta)$ and $\Delta_1(\beta)$. We note that $\Delta(\beta)$ is only conditionally convergent due to the conditional convergence of the product of the principal diagonal whereas $\Delta_1(\beta)$ is absolutely convergent if $\sum_{n=0}^{\infty} f_n$ is absolutely convergent and if none of the. terms blow-up. These last restrictions limit the validity of the dispersion relation to all values of k except $k = mK/\sqrt{\varepsilon}$ where $m = 0,\pm1,\pm2,...$ and limits the periodic variation in ε to that described by the series $f(Kz)$ (see (2.3)) where $\sum_{n=0}^{\infty} f_n$ is absolutely convergent. (This is the technical definition of "physically reasonable" used in Chapter I.)

Now that the limits of validity are established we the definition of the infinite determinant we find by comparing $\Delta(\beta)$ and $\Delta_1(\beta)$ that

$$\Delta(\beta) = \Delta_1(\beta) \lim_{p\to\infty} \prod_{m=-p}^{p} \left\{\frac{(\beta+mK)^2 - k^2\varepsilon}{m^2K^2 - k^2\varepsilon}\right\} \qquad \text{(A.6)}$$

and by expressing $\sin\pi(\beta\pm\sqrt{\varepsilon}k/K)$ and $\sin^2(\pi k/K)$ in product form (A.6) becomes

$$\Delta(\beta) = -\Delta_1(\beta) \; \frac{\sin(\pi(\frac{\beta-k\sqrt{\varepsilon}}{K}))\,\sin(\pi\frac{(\beta+k\sqrt{\varepsilon})}{K})}{\sin^2(\frac{\pi k\sqrt{\varepsilon}}{K})} \qquad \text{(A.7)}$$

For convenience, we define another function $E(\beta)$ in terms of some constant C.

$$E(\beta) \equiv \Delta_1(\beta) + C\left\{\cot(\pi(\frac{\beta+k\sqrt{\varepsilon}}{K})) - \cot(\pi(\frac{\beta-k\sqrt{\varepsilon}}{K}))\right\} \qquad \text{(A.8)}$$

We see that both Δ_1 and $E(\beta)$ are periodic functions of β with period 2K and that $E(\beta)$ has no poles and is bounded. Hence, $E(\beta)$ is equal to a constant by Liouville's theorem. Letting $\beta\to\infty$ we find the constant

$$E(\infty) = \lim_{\beta\to\infty} E(\beta) = 1 \tag{A.9}$$

hence

$$\Delta_1(\beta) = 1 - C\left\{\cot(\pi(\frac{\beta+k\sqrt{\varepsilon}}{K})) - \cot(\pi(\frac{\beta-k\sqrt{\varepsilon}}{K}))\right\} \tag{A.10}$$

Equating $\Delta_1(\beta)$ in (A.10) and (A.8) we find

$$\Delta(\beta) = \frac{-\sin(\frac{\beta-k\sqrt{\varepsilon}}{K})) \sin(\pi(\frac{\beta+k\sqrt{\varepsilon}}{K}))}{\sin^2 (\frac{\pi k\sqrt{\varepsilon}}{K})} + 2C \cot(\frac{\pi k\sqrt{\varepsilon}}{K}) \tag{A.11}$$

Letting $\beta\to 0$ we find

$$\Delta(0) = \lim_{\beta\to 0} \Delta(\beta) = 1 + 2C \cot(\frac{\pi k\sqrt{\varepsilon}}{K}) \tag{A.12}$$

which defines the constant C in terms of $\Delta(0)$. Subtracting (A.12) from (A.11) and using the identity $\sin^2 x-\sin^2 y \equiv \sin(x+y)\sin(x-y)$,

$$\Delta(\beta) - \Delta(0) = \frac{-\sin^2(\pi\beta/K) + \sin^2(\pi k\sqrt{\varepsilon}/K)}{\sin^2(\pi k\sqrt{\varepsilon}/K)} - 1 \tag{A.13}$$

or

$$\Delta(\beta) = \Delta(0) - \frac{\sin^2(\pi\beta/K)}{\sin^2(\pi k\sqrt{\varepsilon}/K)} \tag{A.14}$$

Thus, setting $\Delta(\beta) = 0$ is equivalent to (2.13) or

$$\sin^2(\frac{\pi\beta}{K}) = \Delta(0) \sin^2(\frac{\pi k\sqrt{\varepsilon}}{K}) \qquad \binom{2.13}{A.15}$$

which is the needed result.

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