High-resolution force sensor based on morphology dependent optical resonances of polymeric spheres

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(Received 24 June 2008; accepted 12 November 2008; published online 13 January 2009)

Performance characteristics of a force sensor concept based on the morphology dependent resonance (MDR) shifts of micro-optical resonators have been investigated. Previous experimental studies have indicated that microsphere sensors with diameters ranging between 30 and 950 μm may have force resolutions reaching 10–3 N [T. Ioppolo et al., Appl. Opt. 47, 3009 (2008)]. In the present, we carry out a systematic analysis and experiments to investigate the sensitivity, resolution, and bandwidth limits of MDR-based force sensors. Expressions for MDR shifts due to applied force in the polar direction are obtained for microspheres of various dielectric materials in the diameter range of 300–950 μm. The analyses are compared with experimental results for polymethylmethacrylate and polydimethylsiloxane (PDMS) microsphere sensors. The results show that the strain effect on MDR shifts is dominant over that of mechanical stress. It also indicates that force sensitivities of the order of 1 pN are feasible using hollow PDMS spheres. The sensor bandwidths range between 1 kHz and 1 MHz, depending on the sphere material. © 2009 American Institute of Physics. [DOI: 10.1063/1.3054338]

I. INTRODUCTION

Morphology dependent resonances (MDRs) of dielectric microspheres have attracted interest with proposed applications in a wide range of areas due to the high optical quality factors that they can exhibit. The MDR [also called whispering gallery modes (WGMs)] are optical modes of dielectric cavities such as spheres. These modes can be excited, for example, by coupling light from a tunable laser into the sphere using an optical fiber.1 The modes are observed as sharp dips in the transmission spectrum at the output end of the fiber typically with very high quality factors, $Q = \frac{\lambda}{d\lambda}$ (Q-factors of 1010 are reported for dielectric microspheres) where $\lambda$ is the wavelength of the interrogating laser and $d\lambda$ is the linewidth of the observed mode. The proposed MDR applications include those in spectroscopy,2 microring laser technology,3 and optical communications (switching,4 filtering,5 and wavelength division and multiplexing6). For example, mechanical strain8 and thermo-optical tuning5 of microsphere MDRs have been demonstrated for potential applications in optical switching. Several sensor concepts have also been proposed exploiting the MDR shifts of microspheres for biological applications,9,10 trace gas detection,11 impurity detection in liquids,12 as well as mechanical sensing including force,13 pressure,14 temperature,15 and wall shear stress.1

In mechanical sensing applications, the dielectric microsphere is optically coupled to a single mode fiber which carries light from a tunable laser and serves as an input/output port for the microsphere.13 When the microsphere comes into contact with an exposed section of the fiber core, its optical resonances (MDR) are observed as sharp dips in the transmission spectrum. These optical resonances are extremely narrow (with very high $Q$ values) and thus are highly sensitive to any change in the morphology of the sphere (shape, size, or refractive index). A minute change in the morphology of the microsphere will cause a shift in the resonance positions allowing for the precise measurement of the mechanical effect causing the MDR shift.

Geometric optics provides the simplest interpretation of the MDR phenomenon. The laser light from the optical fiber is coupled to the microsphere nearly tangentially, and approaches the interior surface of the sphere beyond the critical angle, thereby experiencing a total internal reflection along the interior surface of the microsphere. Trapped inside the microsphere, the light circumnavigates the interior surface of the sphere. A resonance (MDR) is realized when light returns to its starting location in phase. Thus, the approximate condition for resonance is

$$2\pi n_0 a = l\lambda,$$ (1)

where $\lambda$ is the vacuum wavelength of laser, $n_0$ and $a$ are the refractive index and the radius of sphere, respectively, and $l$ is an integer indicating the circumferential mode number. Equation (1) is a first order approximation and holds for $a \gg \lambda$. At resonance, light experiences constructive interference in the sphere which is observed as dips in the transmission spectrum through the optical fiber. A minute change in the size of the refractive index of the microsphere will lead to a shift in the resonance wavelength as

$$\frac{d\lambda}{\lambda} = \frac{dn_0}{n_0} + \frac{da}{a}.$$ (2)

In the MDR-based force sensor concept, uniaxial force is applied on the sphere along the polar direction (normal to the plane of light circulation) through two high-stiffness plates.13 The applied force induces a change in both the radius (strain

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effect) and the refractive index (through mechanical stress) of the sphere. In Ref. 13, the MDRs of silica and polymethylmethacrylate (PMMA) spheres have been force tuned, experimentally demonstrating the feasibility of a MDR-based force sensor. The experiments also indicated that force resolutions as high as ~10⁻⁵ N were possible using hollow PMMA spheres.

In the present, an analytical framework is developed for the proposed force sensor. Specifically, expressions are developed for MDR shifts in sphere sensors due to force applied in the polar direction. The analysis accounts for both the strain and stress effects in determining the MDR shifts. Using these expressions, a systemic analysis is carried out to determine the sensitivity, resolution, and bandwidth limits of force sensors based on MDR shifts of polymeric microspheres. The analytical expressions are compared to the experimental results for PMMA and polydimethylsiloxane (PDMS) microsphere sensors. The results show that the strain effect (da/da) dominates over that of mechanical stress (da/da). They also indicate that force sensitivities of the order of a 1 pN are feasible using PDMS spheres.

II. ANALYSIS

A. Stress and strain in a solid dielectric microsphere

We first consider a solid dielectric sphere of radius a that is compressed by two pads (plates) made of high stiffness material (such as stainless steel), as shown in Fig. 1(a). The applied force F will lead to a mechanical stress field inside the microsphere. It will also deform the sphere (strain), as shown in Fig. 1(b), leading to a MDR shift, as depicted in Fig. 1(c). The deformation of the sphere can be obtained by solving the Navier equation:

\[ \nabla^2 u + \frac{1}{1 - 2\nu} \nabla \cdot u = 0, \]

(3)

where \( u \) is the displacement of a given point within the sphere and \( \nu \) is the Poisson ratio. The solution of this equation for an azimuthally symmetric loading is given by\(^\text{16,17}\)

\[ u_r = \sum_{n} \left[ A_n(n + 1)(n^2 - n - 2 + 2\nu)n^3 + B_n(n - 1)r^{n-2}\right] P_n(\cos \theta), \]

(4)

where \( r \) and \( \theta \) are the radial and polar coordinates, respectively [see Fig. 1(a)], \( u_r \) is the radial component of displacement, \( P_n \) is the Legendre polynomial, and \( A_n \) and \( B_n \) are constants determined by the boundary condition at the sphere surface. Using Eq. (4) along with the stress-displacement relationship for an elastic body, the stress distributions within the sphere are obtained as\(^\text{18}\)

\[ \sigma_{rr} = 2G \sum_{n} \left[ A_n(n + 1)(n^2 - n - 2 + 2\nu)n^3 + B_n(n - 1)r^{n-2}\right] P_n(\cos \theta), \]

\[ \sigma_{\theta\theta} = -2G \sum_{n} \left[ A_n(n^2 - 2n + 2\nu + n + 1)r^n + B_n(n - 1)r^{n-2}\right] P_n(\cos \theta), \]

\[ \sigma_{\phi\phi} = 2G \sum_{n} \left[ A_n(n + 1)(n - 2 - 2\nu - 4n)r^n + B_n(n - 1)r^{n-2}\right] P_n(\cos \theta), \]

\[ \sigma_{r\theta} = 2G \sum_{n} \left[ A_n(n^2 + 2n - 1 + 2\nu)r^n + B_n(n - 1)r^{n-2}\right] P_n(\cos \theta). \]

(5)

Here G is the shear modulus of the sphere material. Neglecting the friction at the contact point between the sphere and the plates, the boundary conditions are

\[ \sigma_{r}(a) = \begin{cases} -p(\theta), & 0 \leq \theta \leq \theta_0 \\ \pi - \theta_0 \leq \theta \leq \pi \\ 0, & \theta \leq \pi - \theta_0, \end{cases} \]

(6)

where angle \( \theta_0 \) defines the extent of the contact between the plate and the sphere, as shown in Fig. 1(b). The pressure \( p \) exerted by plates of infinite stiffness on the sphere is given by\(^\text{16,19}\)

\[ p(\theta) = \frac{3F}{2\pi a_0^3} \sqrt{a_0^2 - a^2 \sin^2(\theta)}, \]

\[ \delta = \frac{3F}{2\pi a_0^3} \sqrt{a_0^2 - a^2 \sin^2(\theta)}, \]

and \( a \) is the radius of the microsphere.
\[ a_0 = \left[ \frac{3FR(1 - \nu^2)}{4E} \right]^{1/3}. \]  

Here \( a_0 \) is the radius of contact area, as shown in Fig. 1(b). In order to obtain coefficients \( A_n \) and \( B_n \) in Eqs. (4) and (5), the boundary condition given in Eq. (6) has to be expanded in terms of the Legendre polynomial in the following form: \[ \sigma_{rr}(a) = \sum_n H_n P_n(\cos \vartheta). \]  

Then the coefficient \( H_n \) is obtained as

\[
H_n = (2n + 1) \times \int_0^{\pi/2} P_n(\cos \vartheta) \frac{3F}{2na_0} a_0^3 a^2 \sin^2(\vartheta) \sin d\vartheta.
\]

By satisfying the boundary condition, Eq. (6), for Eqs. (5) and (8), coefficients \( A_n \) and \( B_n \) are obtained as follows:

\[
A_n = \frac{H_n}{2Ga^n[(n + 1)(n^2 - n - 2 - 2\nu) - nn^2 + 2n - 1 + 2\nu]},
\]

\[
B_n = -\frac{H_n}{2G[(n + 1)(n^2 - n - 2 - 2\nu) - nn^2 + 2n - 1 + 2\nu] \times \frac{(n^2 + 2n - 1 + 2\nu)a^{-2n}}{(n - 1)}},
\]

Once coefficients \( A_n \) and \( B_n \) are obtained, strain (displacement) and the stress fields can be calculated by substituting Eq. (10) into Eqs. (4) and (5), respectively.

### B. Force-induced MDR shift in a solid dielectric microsphere

For the solid microsphere, the last term in Eq. (2) can be calculated by evaluating Eq. (4) at \( r = a \) and then dividing it by the sphere radius \( a \):

\[
\frac{da}{a} = \sum_n \frac{H_n}{4G(n^2 + 2n\nu + n + \nu + 1)} \times \left[ \frac{2 - 4n^2 + \nu(4n^2 - 2n - 4)}{n - 1} \right] P_n(0). \tag{11}
\]

Next we determine the effect of stress on refractive index perturbation, \( dn_0/n_0 \), in Eq. (2). Neumann–Maxwell equations provide the relationship between stress and refractive index as follows:\[ n_r = n_{0r} + C_1 \sigma_{rr} + C_2 (\sigma_{\theta\theta} + \sigma_{\phi\phi}) \]
\[ n_\theta = n_{0\theta} + C_1 \sigma_{\theta\theta} + C_2 (\sigma_{rr} + \sigma_{\phi\phi}) \]
\[ n_\phi = n_{0\phi} + C_1 \sigma_{\phi\phi} + C_2 (\sigma_{\theta\theta} + \sigma_{rr}) \]

Here \( n_r, n_\theta, n_\phi \) are the refractive indices in the direction of the three principle stresses and \( n_{0r}, n_{0\theta}, n_{0\phi} \) are those values for the unstressed material. Coefficients \( C_1 \) and \( C_2 \) are the elasto-optical constants of the material, and for both PMMA and PDMS, \( C_1 = C_2 \). For PMMA \( C_1 = C_2 = C = 10^{-10} \text{ m}^2/\text{N} \) (Ref. 21) and for PDMS this value is \( C_1 = C_2 = C = -1.75 \times 10^{-10} \text{ m}^2/\text{N} \). Thus, for a spherical sensor, the fractional change in the refractive index due to mechanical stress is reduced to

\[
\frac{dn_0}{n_0} = \frac{n_r - n_{0r}}{n_{0r}} = \frac{n_\theta - n_{0\theta}}{n_{0\theta}} = \frac{n_\phi - n_{0\phi}}{n_{0\phi}} = \frac{C(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{\phi\phi})}{n}.
\]

In the present MDR optical sensor, light is traveling in a plane that is normal to the applied force. Thus, evaluating the appropriate expressions for stress in Eq. (5) at \( \vartheta = \pi/2 \) and \( r = a \), and introducing them into Eq. (13), the relative change in the refractive index due to force \( F \) is obtained as

\[
\frac{dn_0}{n_0} = C \sum_n \left( \frac{H_n(n + 1)(n^2 + 3n + 4 + 4\nu + 4m)}{(n^2 + 2m + n + \nu + 1)} \right) P_n(0). \tag{14}
\]

Equations (11) and (14) represent the effect of strain and stress, respectively, on the MDR shift of the solid dielectric sphere. Plugging these into Eq. (2), we obtain the total MDR shift as

\[
\frac{d\lambda}{\lambda} = \sum_n \frac{H_n}{2(n^2 + 2n\nu + n + \nu + 1)} \times \left[ \frac{2 - 4n^2 + \nu(4n^2 - 2n - 4)}{n - 1} \right] P_n(0). \tag{15}
\]

### C. Stress and strain in a hollow dielectric microsphere

In this case we consider a hollow dielectric microsphere with outer and inner radii of \( a \) and \( b \), respectively. Again, in order to obtain the MDR shift induced by the applied force, the strain and stress at the sphere’s outer surface must be known. To obtain the strain and stress distributions for a hollow microsphere, we superimpose the solution of a solid microsphere [Eqs. (4) and (5)] and that of a hollow cavity in an infinite medium.\[ u_r = \sum n C n(n + 3 - 4\nu - D_n(n + 1)) P_n(\cos \vartheta), \]  

and the corresponding stress field is given by\[ \sigma_{rr} = 2G \sum \left( \frac{C_n}{n^{\nu+2}}(n^2 + 3n - 2\nu) \right) \frac{D_n(n + 1)(n + 2)}{n^{\nu+3}} P_n(\cos \vartheta). \]
\[ \sigma_{\theta \theta} = 2G \sum_{n=1}^{\infty} \left[ \frac{C_n}{r^{n+1}} (n^2 - 2n - 1 + 2\nu) - \frac{D_n(n+1)^2}{r^{n+3}} \right] \times P_n(\cos \theta) - \left[ \frac{C_n}{r^{n+1}} (-n + 4 - 4\nu + D_n) \right] \times \cot \theta \frac{dP_n(\cos \theta)}{d\theta}, \]

\[ \sigma_{\phi \phi} = 2G \sum_{n=1}^{\infty} \left[ \frac{C_n}{r^{n+1}} (n + 3 - 4n\nu - 2\nu) - \frac{D_n(n+1)}{r^{n+3}} \right] \times P_n(\cos \theta) + \left[ \frac{C_n}{r^{n+1}} (-n + 4 - 4\nu + D_n) \right] \times \cot \theta \frac{dP_n(\cos \theta)}{d\theta}, \]

\[ \sigma_{r \theta} = 2G \sum_{n=1}^{\infty} \left[ \frac{C_n}{r^{n+1}} (n^2 - 2 + 2\nu) - \frac{D_n(n+2)}{r^{n+3}} \right] \times dP_n(\cos \theta) \frac{d}{d\theta}. \] (17)

The strain distribution in a hollow sphere is obtained by adding Eqs. (4) and (16). Likewise, the stress distribution is obtained by adding Eqs. (5) and (17).

The coefficients \( A_n, B_n, C_n, \) and \( D_n \) are determined by satisfying the boundary condition at the inner and outer surfaces of the hollow microsphere. Again at the outer surface we assume that the friction at the contact surface between the sphere and the plates is negligible. We also assume that the sphere wall is thick enough such that when the force \( F \) is applied to the sphere, the pressure inside the sphere cavity remains constant. With these assumptions, the boundary conditions are

\[ \sigma_{\theta \theta}(a) = \begin{cases} -p(\theta), & 0 \leq \theta \leq \theta_0 \text{ and } \pi - \theta_0 \leq \theta \leq \pi, \\ 0, & \theta_0 \leq \theta \leq \pi - \theta_0, \end{cases} \]

\[ \sigma_{r \theta}(a) = 0 \quad 0 \leq \theta \leq \pi, \]

\[ \sigma_{r \theta}(b) = 0 \quad 0 \leq \theta \leq \pi, \]

\[ \sigma_{r \theta}(b) = 0 \quad 0 \leq \theta \leq \pi. \] (18)

For the hollow sphere solution, the superimposition of Eqs. (5) and (17) leads to the following linear system:

\[ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} A_n \\ B_n \\ C_n \\ D_n \end{pmatrix} = \begin{pmatrix} 0 \\ H_n \end{pmatrix}, \] (19)

where coefficients \( a_{ij} \) are determined from the boundary conditions of Eq. (18). Once the coefficients \( A_n, B_n, C_n, \) and \( D_n \) are known, the strain and stress components can be evaluated.

D. Force-induced MDR shift in a hollow dielectric microsphere

The resonance shift for the hollow microsphere is obtained using the same procedure followed for the solid microsphere:

\[ \frac{d\lambda}{\lambda} = \frac{da}{a} + \frac{dn_0}{n_0} \]

\[ = \sum_n \left[ A_n(n+1)(n-2+4\nu)a^n + B_n(n+1)a^{n-1} \right] \left( \frac{C_n}{a^n} \right) \left( \frac{D_n}{a^{n+1}} \right) P_n(0) \]

\[ + \sum_{n_0} \left[ -A_n(n+1)(n^2+3n+4n\nu+4\nu+4)a^n \right] \frac{C_n}{a^{n+1}} P_n(0) \]

\[ + \sum_{n_0} \left[ -A_n(n^2-4n\nu+2) \frac{D_n}{a^{n+3}} P_n(0) \right]. \] (20)

Here the first term on the right hand side of Eq. (20) represents the strain effect and the second term represents that of stress.

Figure 2 shows the MDR shifts of the solid and hollow PMMA spheres \((D=400 \mu\text{m})\) under compressive force. The effect of force on \( da/a \) (strain) and \( dn_0/n_0 \) (stress) are shown separately to compare the contribution of each of these terms on the MDR shift, \( d\lambda/\lambda \). The figure shows that strain effect is dominant over that of stress. The results of Fig. 2 also indicate that for the force range considered, the dependence of \( d\lambda/\lambda \) on force is essentially linear.

III. EXPERIMENTS

The analytical expressions developed in Sec. II are validated through experiments carried out for both PMMA (of Ref. 13) and PDMS spheres (current experiments). In Ref. 13, in addition to solid silica spheres, solid and hollow PMMA spheres were used to experimentally investigate the dependence of MDR shifts on uniaxial force. Those results showed that the force sensitivity of solid PMMA spheres...
were significantly higher than their silica counterparts due to the smaller Young modulus of PMMA as compared to silica. The Young modulus for silica is \( \sim 70 \text{ GPa} \) whereas this value for PMMA is \( \sim 3.3 \text{ GPa} \). (Note that we define force sensitivity as \( \Delta \lambda /dF \) where \( dF \) is the applied incremental force). The experiments of \(^{13}\) showed that sensitivity is further improved by using hollow spheres (leading to a force resolution of \( \sim 10^{-5} \text{ N} \)).

In the presented study, we use PDMS spheres to further increase force sensitivity. PDMS is an elastomer that is a mixture of two components: a base and a curing agent. By changing the ratio between the base and the curing agent, it is possible to obtain elastic materials with Young’s moduli ranging between 3 and 1000 kPa (corresponding to a base-to-curing agent volume ratio of 60:1 and 10:1, respectively).

The optoelectronic setup for the currently presented force sensitivity studies is essentially the same as that used in Ref. 13. The setup is shown schematically in Fig. 3. Briefly, a tunable distributed feedback laser (with 1.312 \( \mu \text{m} \) central wavelength and 10 mW maximum power) drives the system. It is current tuned over a range of \( \sim 0.1 \text{ nm} \). The laser is coupled into a single mode optical fiber, whose output end was connected to a fast photodiode to monitor the transmission spectrum. A section of the fiber is tapered down to a diameter of \( \sim 10 \mu \text{m} \) (by heating and stretching the fiber). The microsphere was brought into contact with the tapered fiber section to facilitate light coupling between fiber and resonator. The transmission spectrum through the fiber is normalized by a reference signal taken directly from the laser. The transmission spectrum is digitized using a 16 bit analog to digital converter and the MDR positions are determined using an in-house software. A personal computer controls both the frequency tuning of the laser and the data acquisition.

### A. PMMA sensor

In Fig. 4, the experimental results for solid and hollow PMMA spheres are compared to the analytical results. For this calculation, the Young modulus for PMMA was taken as \( E=3.3 \text{ GPa} \). The diameter of the solid PMMA sphere is \( \sim 470 \mu \text{m} \) and the diameter of the hollow sphere is \( \sim 960 \mu \text{m} \) with a wall thickness of about 20 \( \mu \text{m} \). The figure shows a good agreement between experimental results of \(^{13}\) and the presented analytical predictions for both solid and hollow spheres.

### B. PDMS sensor

The PDMS microspheres were manufactured by dipping the tip of a silica fiber inside a freshly mixed PDMS polymer pool. A number of PDMS microspheres were manufactured using different mixture ratios, thereby having different Young’s moduli. The Young modulus of a PDMS microsphere is substantially smaller than that of PMMA sphere of the same diameter resulting in comparable sphere deformations at much smaller force levels. Thus, determining the sensitivity of the PDMS force sensors is a more challenging task; the load-cell based measurement setup of \(^{13}\) would not provide sufficient resolution for sensor calibration. Moreover, since the force levels in these calibrations are \( (<10^{-6} \text{ N}) \), even the aerodynamic forces due to air currents (air drafts and natural convection) would have considerable adverse effect. To avoid such problems, the setup shown in Fig. 5 has been built and kept enclosed in a vacuum chamber, as shown in the figure. The measurements were carried out at \( \sim 50 \text{ Torr} \) (\( \sim 65 \text{ mbar} \)) to reduce the effects of aerodynamic forces. A cantilever beam is used to apply force on the spheres. The beam is a silica fiber \( (E=75.8 \text{ GPa}) \) with a length and diameter of 135 and 0.635 mm, respectively. The beam was...
clamped onto a microtranslation stage on one end, while the other end (which was melted and flattened to have a flat plate) was in contact with the microsphere. Moving the translation stage toward the sphere bends the cantilever beam thereby exerting force on the sphere. In order to accurately measure the displacement of the translation stage, a Michelson interferometer was assembled, as shown in Fig. 5. Note that the deflection of the cantilever beam is <3 μm; therefore, the geometry of the contact region between the sphere and cantilever remains essentially the same during the loading process. During each set of measurements, the temperature of the chamber was kept constant to avoid temperature-induced MDR shifts.\textsuperscript{14}

As pointed out above, the Young modulus of the PDMS is significantly lower, leading to larger sphere deformations. For some cases studied, the sphere deformation could even be comparable or larger than the deflection of the cantilever beam. Therefore, the displacement of the translation stage (at the base of the beam) registered by the interferometer is the sum of the beam deflection and sphere deformation which makes the determination of the force exerted on the sphere ambiguous. In order to account for the sphere deformation, a Hertz-contact analysis\textsuperscript{16} was carried out on the spheres for a range of force values. Using this force versus deformation data and modeling the sphere as a linear spring system, the equivalent spring constant \( k_{\text{sphere}} \) was determined for each sphere size and PDMS mixture ratio used in the experiments. Then, the equivalent spring constant of the beam-sphere system was determined as

\[
\frac{1}{k_e} = \frac{1}{k_{\text{sphere}}} + \frac{1}{k_{\text{beam}}},
\]

where the spring constant for the beam is

\[
k_{\text{beam}} = \frac{3E\pi D^4}{64L^3}.
\]

Here, \( E \) is the Young modulus, \( D \) is the diameter, and \( L \) is the length of the silica beam. The force exerted on the sphere is then calculated as

\[
F = k_{\text{beam}} \delta,
\]

where \( \delta \) is the displacement of the translation stage and is determined by the Michaelson interferometer signal as

\[
\delta = \frac{l\lambda_m}{2}.
\]

Here, \( l \) is an integer and \( \lambda_m \) is the wavelength of the He–Ne laser (632.8 nm).

Figure 6 shows the MDR shift dependence on the applied force for a solid PDMS microsphere of 910 μm diameter with base-to-curing agent volume ratio of 50:1 (\( E = 10 \) kPa). The figure indicates a strong agreement between the experiments and Eq. (15).

**IV. FORCE SENSITIVITY AND RESOLUTION**

Figure 6 indicates that for PDMS spheres, as in the case of the other materials used previously,\textsuperscript{13} the relationship between force and MDR shift is essentially linear in the force range studied. The slope of the MDR shift (sensitivity) \( d\Delta/dF \) is a function of the microsphere material and geometry (size and whether it is a solid or hollow microsphere). Thus, by choosing different materials and varying sphere geometry, a wide range of sensitivities can be obtained. Using the analyses of Secs. II A and II B, the sensitivity of PMMA and PDMS microspheres are investigated systematically. The sensitivity of hollow PMMA (\( E = 3.3 \) GPa) sensors with varying sphere diameters \( D \) and inner-to-outer radius ratios, \( b/a \) is shown in Fig. 7(a). The figure indicates that for very thin walled spheres (\( b/a > 0.9 \)), the sphere behaves like a membrane and for a given \( b/a \) value the sensitivity is essentially independent of the sphere diameter. The data of Fig. 7(a) can be represented by the following approximate equation:

**FIG. 6.** Comparison between experimental and analytical results for a solid PDMS (50:1) microsphere with \( D = 910 \) μm.

**FIG. 7.** Sensitivity curves for microspheres of different types: (a) hollow PMMA and (b) hollow PDMS (400 μm diameter).
TABLE I. The constants of Eq. (25) for solid and hollow PMMA spheres.

<table>
<thead>
<tr>
<th></th>
<th>K</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solid PMMA</td>
<td>$2.217 \times 10^4$</td>
<td>5.74</td>
</tr>
<tr>
<td>Hollow PMMA (b/a&lt;0.91)</td>
<td>$9.687 \times 10^4$</td>
<td>3.54</td>
</tr>
<tr>
<td>Hollow PMMA (b/a&gt;0.91)</td>
<td>0.0362</td>
<td>20</td>
</tr>
</tbody>
</table>

where the values of constants $K$ and $R$ depend on the sphere geometry and are given in Table I. Both $K$ and $R$ are calculated by curve fitting in sensitivity plots. Figure 7(b) shows the effect of Young’s modulus, $E$ (for a range corresponding to that for PDMS), and $b/a$ on sensor sensitivity $d\lambda/dF$. The data of Fig. 7(b) can be expressed by the approximate equation

$$\frac{d\lambda}{dF} = KD^{-2}e^{R(b/a)},$$

(25)

where $K=6.884 \times 10^4$ and $R=3.3$ for $b/a<0.91$ and $K =1.744 \times 10^2$ and $R=22$ for $b/a>0.9$. A comparison of Figs. 7(a) and 7(b) shows that PDMS sensors can provide significantly higher sensitivities than PMMA counterparts.

Figure 8 shows the influence of sphere diameter $D$ on sensitivity of solid PDMS sensors as a function of Young’s modulus $E$. The results of Fig. 8 indicate that the sensitivity for the solid sphere can be expressed as

$$\frac{d\lambda}{dF} = 8.388 \times 10^8 D^{-2} E^{-1}$$

(26)

As expected, a decrease in both the sphere diameter and Young’s modulus results in increasing the sensitivity. Together with the sensitivity $d\lambda/dF$, the optical quality factor $Q\lambda/d\lambda$ determines the minimum measurement resolution. If we assume that the minimum measurable MDR shift is $\Delta\lambda=\lambda/Q$, the measurement resolution is

$$\delta F = \frac{\lambda}{Q} \left( \frac{d\lambda}{dF} \right)^{-1}.$$

(28)

Figure 8 indicates that for a solid PDMS sensor of ~900 $\mu$m diameter and a mixture ratio of 50:1, a force resolution of ~10 nN is possible with a $Q$-factor of 10. Sensitivities and force resolutions of several PMMA and PDMS spheres have been listed in Table II. The data listed for PMMA are the results from previous experimental studies, whereas the data for PDMS are calculated analytically. As it is indicated in the table, choosing hollow geometry for the polymeric microspheres may improve the sensitivity around two orders of magnitude. We note that these results also indicate that with hollow, 60:1 mixture ratio PDMS sensors, force resolutions of the order of a piconewton may be feasible.

V. SENSOR BANDWIDTH

Sensor bandwidth is an important parameter in most force measurements. We take the highest frequency response of the sensor as the first natural frequency of the sphere. In this approach, the other moving parts of the sensor are not taken into account and the sphere is considered undamped. It should be noted that the additional mass of the other moving parts will reduce the bandwidth estimate. We calculate the resonant frequency by numerically solving the characteristic equation for the sphere

$$1 - \nu = \frac{1}{2} \left( a \frac{\omega_n}{C_F} J_{1/2}\left(a \frac{\omega_n}{C_F}\right) - 2 a \frac{\omega_n}{C_F} J_{3/2}\left(a \frac{\omega_n}{C_F}\right) \right) 
= 0,$$

(29)

where $\omega_n$ is the angular frequency, $J_{n}'s$ are the Bessel functions of first kind, and $C_F$ is the compressive wave velocity defined as

$$C_F = \sqrt{\frac{\lambda + 2 \mu}{\rho}},$$

(30)

where $\lambda$ and $\mu$ are the Lamé constants and $\rho$ is the microsphere material density. The results for solid PDMS sphere sensors are shown in Fig. 9. Again, the values in this figure represent the highest possible frequency responses that are associated with the force sensor. They do not take into account any other moving parts of these sensors other than the sphere itself. In Sec. IV, it was shown that the lower the Young modulus, the better the force resolution. However, there is a trade-off. As it is seen in the figure, the sensor bandwidth is significantly smaller for sphere sensors with...
PMMA, shown to have better force resolutions than those of the solid on bandwidth. Also, while hollow PMMA spheres have been shell sensors by the compressibility of the gas inside the shell. Thus, for crosphere defined as sensors of higher sensitivity. Note that for hollow spheres with PMMA as well as hollow PDMS spheres are shown in Fig. 1100
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Young’s moduli than those with the higher values. We also note that for a given E, the smaller the sensor, the larger the bandwidth due to the effect of sensor mass. In the case of hollow microsphere, a thin shell filled with air is considered. The resonant (frequency) modes for the case (b/a > 0.91) are obtained by solving the equation

\[
\left[ \frac{1-v}{1+v} \right] \beta \left( \frac{\omega R}{c} \right)^2 \left[ 1 + \beta \left( \frac{\omega R}{c} \right)^2 \right] + n(n+1)
\]

\[
\times \left[ 1 - \frac{\beta \left( \frac{\omega R}{c} \right)^2}{1-v} \right] \left[ \frac{J_{n+\frac{1}{2}}(\omega R/c)}{J_n(\omega R/c)} \right]
\]

\[
+ \frac{\rho}{\rho_s} R \frac{R^2}{2} \left( 1 - \frac{v}{1+v} \right) \left( 1 + \beta \left( \frac{\omega R}{c} \right)^2 \right)
\]

\[
\frac{n(n+1)}{1+v} = 0,
\]

where \(J_n\) are the Bessel functions of first kind, \(c\) is the speed of the sound of air in sphere shell, \(R\) is the average radius (average of inner and outer radius), and \(h = a - b\), \(\rho_s\), and \(\rho\) are the density of the shell material and the fluid inside the microsphere (air for the present case) respectively, and \(\beta\) is defined as

\[
\beta = \frac{\rho_s c^2}{E} (1 + v).
\]

The sensor upper bandwidth limits for a solid and hollow PMMA as well as hollow PDMS spheres are shown in Fig. 10. As expected, the sensor bandwidth is narrower for sensors of higher sensitivity. Note that for hollow spheres with \(b/a > 0.91\), the system is essentially a shell and the resonant frequencies of the system (sensor bandwidth) is dominated by the compressibility of the gas inside the shell. Thus, for shell sensors \(b/a\) and \(E\) do not have any significant influence on bandwidth. Also, while hollow PMMA spheres have been shown to have better force resolutions than those of the solid PMMA, they have narrower sensor bandwidth when compared with their solid counterpart. For PDMS microsphere sensors, however, the solution of Eq. (31) for sphere shells indicate better sensor bandwidths than those of the solid ones predicted by Eq. (29). This is due to the small mass of the thin walled sphere shell as compared to the solid sphere. Thus, hollow PDMS microsphere sensors provide both high force resolutions and higher bandwidths.

VI. CONCLUSION

Analytical studies of polymeric spheres under the effect of uniaxial compressive force have been carried out and the analysis have been validated through experiments. The results show that the force sensitivity for a MDR sensor is a function of the sphere material property and geometry. A wide range of sensitivities can be obtained by choosing different combinations of sphere material and geometry. Force resolutions of the order of nanonewtons have been observed experimentally, while analytical results show that hollow sphere sensors made of PDMS may reach up to piconewton resolution. While the sensitivity can be improved by choosing a material with lower Young’s modulus, there is a trade-off in performance. For solid polymeric microspheres, the bandwidth of the sensor becomes smaller as the sensitivity increases. For PMMA and PDMS sensors, the bandwidths are calculated to be of the order of megahertz and hundred kilohertz, respectively. The hollow PMMA microspheres also have narrower bandwidth compared with its solid counterpart. For PDMS microsphere sensors, however, the analysis indicates that thin walled hollow PDMS sensors have both higher force resolutions and larger bandwidths than their solid counterparts, making them highly attractive in sensor applications.


FIG. 9. Estimated sensor bandwidths for solid PDMS spheres.

FIG. 10. Estimated sensor bandwidth for solid PMMA, and hollow PDMS and hollow PMMA.
14, 790 (1997).