TIME-PHASED REAL ESTATE DEVELOPMENT: AN INTEGER PROGRAMMING MODEL

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1.0 Introduction.

This real estate development problem is an expansion and refinement of a problem suggested by Dr. Richard B. Peiser, Assistant Professor of Real Estate and Regional Science at Southern Methodist University. The ensuing integer programming model was originally developed but never solved by Mark D. Youtsey, a senior engineering student at SMU, in 1980. The following model, however, has been further developed to reflect current costs, interest rates, inflation rates, and a clearer representation of annual cash flows.

This integer programming model is a mathematical investment analysis used to determine optimal investment decisions in the development of real estate. It utilizes a modified version of linear programming to accommodate integer (0,1) variables for an integer programming solution.

2.0 The Problem.

The problem addressed by this model is described as follows:
A real estate developer/investor owning a large tract of land wishes to determine the optimal method of developing and selling various plots of this land so as to maximize profit. The development project is to extend over a 5-year period. The land is divided into four zones of different sizes or acreages (see Figure A). Each zone is further apportioned into three types according to use or purpose: (1) residential, (2) industrial, and (3) commercial. That **is**, each zone has a different allotment of land suitable for each of these three purposes.

FIGURE A

LAND APPORTIONMENT

(in acres)

ZONE 1

Tvpe	Allocation
1	0
2	150
3	150
Total	300

ZONE 2

Type	Allocation
1	300
2	100
3	0
Total	400

ZONE 3

Type	Allocation,
1	375
2	0
3	25
Total	400

ZONE 4

Type	Allocation
1	475
2	100
3	25
Total	600

1 = Residential

2 = Industrial

3 = Commercial

The developer is faced with fixed (startup) costs associated with opening a zone for development, as well as marginal or variable costs for developing each type of land within the zone (see Table 1). Development of the land is stimulated by annual demand and revenues from selling the developed land. In order to finance these developments the investor is given a yearly line of credit of $5 million at 18 per cent interest per annum. A one-year loan may be procured at the beginning of each year up to the credit limit and must be repaid with interest at the end of that year (no compounding). All selling prices and costs are subject to an annual inflation rate of 10 per cent.

3.0 Assumptions.

The assumptions which have been incorporated into this model are:

(1) Demand for land does not vary with development (constant demand).*

(2) Price does not vary with demand (constant price with respect to demand).*

(3) Once a zone is opened for development any and all parts of that zone are eligible for development and subsequent sale.

4.0 Description of the Model.

This model is a large integer programming model. The formulation is comprised of a maximum profit objective function in 160 integer and continuous (linear) variables, subject to 160 constraints.

The maximum profit sought in this model is the result of revenues from land sales and interest onn cash less the costs of land development and interest

* These assumptions, although contradictory to economic theory, are necessary to enable this model to take on its integer characteristics, as opposed to the- more complex, dynamic programming model.

TABLE 1

* COSTS:

I. Fixed Startup Costs:

Zone	Cost (millions)
1	$1.50
2	$1.75
3	$2.00
4	$2.25

II. Variable Costs:

Land Type	Cost
1	**$10,000**
2	$ 6,000
3	$ 6,000

* PRICES:

I. Selling Prices for Land

Type	Cost Per Acre
1	**$30,000**
2	$44,000
3	$04 000

DEMAND: (in acres)

Type \ Year	1	2	3	4	5
1	100	200	300	300	300
2	25	50	75	75	75
3	0	50	50	50	50

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* An inflation rate of 8° per annum is built into each year's costs and prices. For each of the 5 operating years, this is reflected in the coefficients of the decision variables.

on loans. The profitability of this endeavor is constrained by:

(1) land availability

(2) land apportionment

(3) demand for developed land

(4) capital availability (due to credit limits)

(5) the fact that land may not be sold until it is developed..

5.0 Definition of Decision Variables and Coefficients.

The decision variables and coefficients for this model are as follows:

LS_{ijt} = amount of land sold of development type "j" in zone "i" during year "t" (in acres),

where

i = 1,2,3,4 (zone numbers)

j = 1 land for residential development
2 " " industrial "
3 " " commercial "

t = 1,2,3,4,5 (years of operation and investment).

LD_{ijt} amount of land developed of type "j" in zone "i" during year "t" (in acres).

LA_{ij} amount of land in zone "i" allocated for development type "j".

D_{jt} demand for land of type "j" during year "t".

L_t loan procured during year "t" for investment toward the development of land.

C_t idle cash on deposit in a bank during year "t" (gaining interest of 5.25% annually while on deposit).

r_t interest rate on any loan procured at the beginning of year "t". 18% per annum.

b_{it} 0 if zone "i" is not developed in year "t" (not opened)
1 if zone "i" is opened for development in year "t".

s_{it} fixed startup cost for developing zone "i" in year "t".

P_{jt} selling price of land type "j" in year "t".

M_{jt} marginal (variable) cost of developing land type "j" in year "t".

$(CASHIN)_t$ = sum of all cash in-flows during year "t".

$(CASHOUT)_t$ = sum of all cash out-flows during year "t".

6.0 The Integer Programing Model.

This real estate development problem may be formulated as the following integer programming model:

Objective Function:

Maximize Profit = Z

$$= \sum_{t=1}^{5} \sum_{i=1}^{4} \sum_{j=1}^{3} (LS_{ijt} P_{jt} - LD_{ijt} M_{jt}) - b_{it} S_{it} - \sum_{t=1}^{5} r L_t$$

where i = 1,2,3,4
j = 1,2,3
t = 1,2,3,4,5

subject to:

(1) $\sum_{t=1}^{5} LS_{ijt} \leq \sum_{t=1}^{5} LD_{ijt}$

(2) $LS_{i1t} \leq D_{1t}$ (j = 1, residential use)

$LS_{i2t} \leq D_{2t}$ (j = 2, industrial use)

$LS_{i3t} \leq D_{3t}$ (j = 3, commercial use).

(3) $\sum LD_{ijk} \leq A_{ij} (\sum S_{ik})$

(4) $\sum_{t=1}^{5} S_{it} = 1$

were $S_{it} = (0,1)$

(5) $(CASHIN)_t = \sum LS_{ij(t-1)} P_{ij(t-1)} + 1.0525\, C_{t-1} + L_t$

(6) $(CASHOUT)_t = \sum LD_{ijt} M_{jt} + \sum s_{it} S_{it} + (1 + r_{t-1}) L_{t-1}$

(7) $(CASHIN)_t = (CASHOUT)_t$

(8) $L_t \leq 5{,}000{,}000$

7.0 The Solution Phase.

7.1 The Relaxed Solution. This model, once formulated, was solved using the MINOS Software System for large-scale linear programs. Due to the fact that the MINOS System solves only continuous problems, the model was entered as a linear program (allowing (0,1) variables to take on positive fractional values summing to unity) and the "relaxed" solution resulted. The optimal relaxed solution resulted in a maximum profit over the 5-year period of

$50,623 ,,805.00

7.2 The Integer Solution. After obtaining the relaxed solution, a makeshift branch-and-bound analysis was done by "turning on and off" the various integer variables for each zone. This was achieved by fixing "delta.' values at 1 to open and close the development of zones during the various time periods, since a zone can not be partially open for development as the relaxed solution would suggest. Upon examination of all possible combinations of zone openings and closings, disregarding any openings which resulted in infeasible solutions, the following optimal integer, solution resulted.

7.2.1 Zone Openings.

Year 1: Zones 1 and 3 open for development.

Year 3: Zones 2 and 4 open for development.

7.2.2 Land Sales and Development. The land sales figures resulting in maximum profit are found in Table 2, while the specific figures on land development timing and acreages are located in Table 3.

7.2.3 Profit. The total maximum profit derived from the optimum integer solution is

$50,598,845.00

a decrease in total profit of $24,960.00 or 0.05% from the relaxed solution.

This decrease in profit from the relaxed to integer solution is the result of the integer (0,1) variables being attached to fixed costs. The relaxed, continuous solution enabled the integer variables to take on positive fractional values. With this kind of flexibility of solution, the relaxed formulation is able to better minimize the costs which are deducted from the total profit.

7.2.4 Loans. The loans procured by the investor for maximum profit include a $5 million loan at the beginning of year 1 (full credit limit), and a $2.92 million loan at the beginning of year 2 in order to finance the development operations. No other loans are necessary, since the profits derived from the first 2 years of operation provide sufficient capital for the remaining 3 years of operation.

7.2.5 Cash Flow. The annual cash flows (in and out) resulting from the 5 years of operational revenues and expenditures are:

Year.	Cash Flow
1	$5 million
2	$5.5 million
3	$12.6 million
4	$18.8 million
5	$36.5 million

The excess of revenues over expenses produces the optimal $50,598,845 profit.

8.0 Conclusion.

Based upon this analysis and the resulting solution, one may conclude that this 5-year endeavor is quite profitable. However, more generally this problem simply illustrates the value of this model in an investment analysis. Given the appropriate information (cost figures, demand, prices) to a real investment problem, this model and formulation could be easily adapted and used as a means of discerning the optimum development alternatives and profitability of potential real estate, portfolio, or other capital investments.

TABLE 2

LAND SALES SCHEDULE

Year	Zone	Type	Acres Sold	Period Totals
1	1	2	25	
	3	1	50	75
2	1	2	50	
		3	25	
	3	1	200	
		3	25	300
3	1	2	75	
		3	25	
	4	1	300	
		3	25	425
4	1	3	50	
	2	1	125	
		2	25	
	4	1	175	
		2	50	425
5	1	3	50	
	2	1	175	
		2	75'	
	3	1	125	425
Total				1650

TABLE 3

LAND DEVELOPMENT SCHEDULE

Type	Zone	Type	Acres Developed	Period Totals,
1	1	2	25	
	3	1	120	
		3	25	170
2	1	2	125	
		3	150	
	3	1	255	530
3	2	1	47.5	
		2	100	
	4	1	475	
		3	25	647.5
4	2	1	252.5	
	4	2	50	302.5
5			0	0
Total				1650