EE 5345
FALL '96
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SOLUTIONS: DSP HOMEWORK

1. $x[n-m] \longleftrightarrow z^{-m} X(z)$

$$\sum_{n=-\infty}^{\infty} x[n-m] z^{-n} \quad , \quad \ell = n-m, \quad n = \ell + m$$

$$\sum_{\ell=-\infty}^{\infty} x[\ell] z^{-\ell} \cdot z^{-m} = X(z) z^{-m} \quad \square$$

$$x[n] * y[n] \longleftrightarrow X(z) Y(z)$$

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m] y[n-m] z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} x[m] \underbrace{\sum_{n=-\infty}^{\infty} y[n-m] z^{-n}}_{z^{-m} Y(z) \text{ by previous property}}$$

$$= X(z) Y(z) \quad \square$$

2. $h[n] = \{\underset{\uparrow n=0}{2}, -1, 1\}$, $x[n] = \{\overset{n=0 \downarrow}{2}, 3, 5, 1, 6, 3\}$

1, −1, 2 ⟶ n

TIME DOMAIN

$y[0] = 2 \cdot 2 = 4$
$y[1] = 3 \cdot 2 + 2(-1) = 4$
$y[2] = 5 \cdot 2 - 3 + 2 = 9$
$y[3] = 1 \cdot 2 - 5 + 3 = 0$
$y[4] = 6 \cdot 2 - 1 + 5 = 16$
$y[5] = 3 \cdot 2 - 6 + 1 = 1$
$y[6] = -3 + 6 = 3$
$y[7] = 3 = 3$

$y[n] = \{\underset{\uparrow n=0}{4}, 4, 9, 0, 16, 1, 3, 3\}$

Z-DOMAIN:

$X(z)$: $2+3z^{-1}+5z^{-2}+1z^{-3}+6z^{-4}+3z^{-5}$

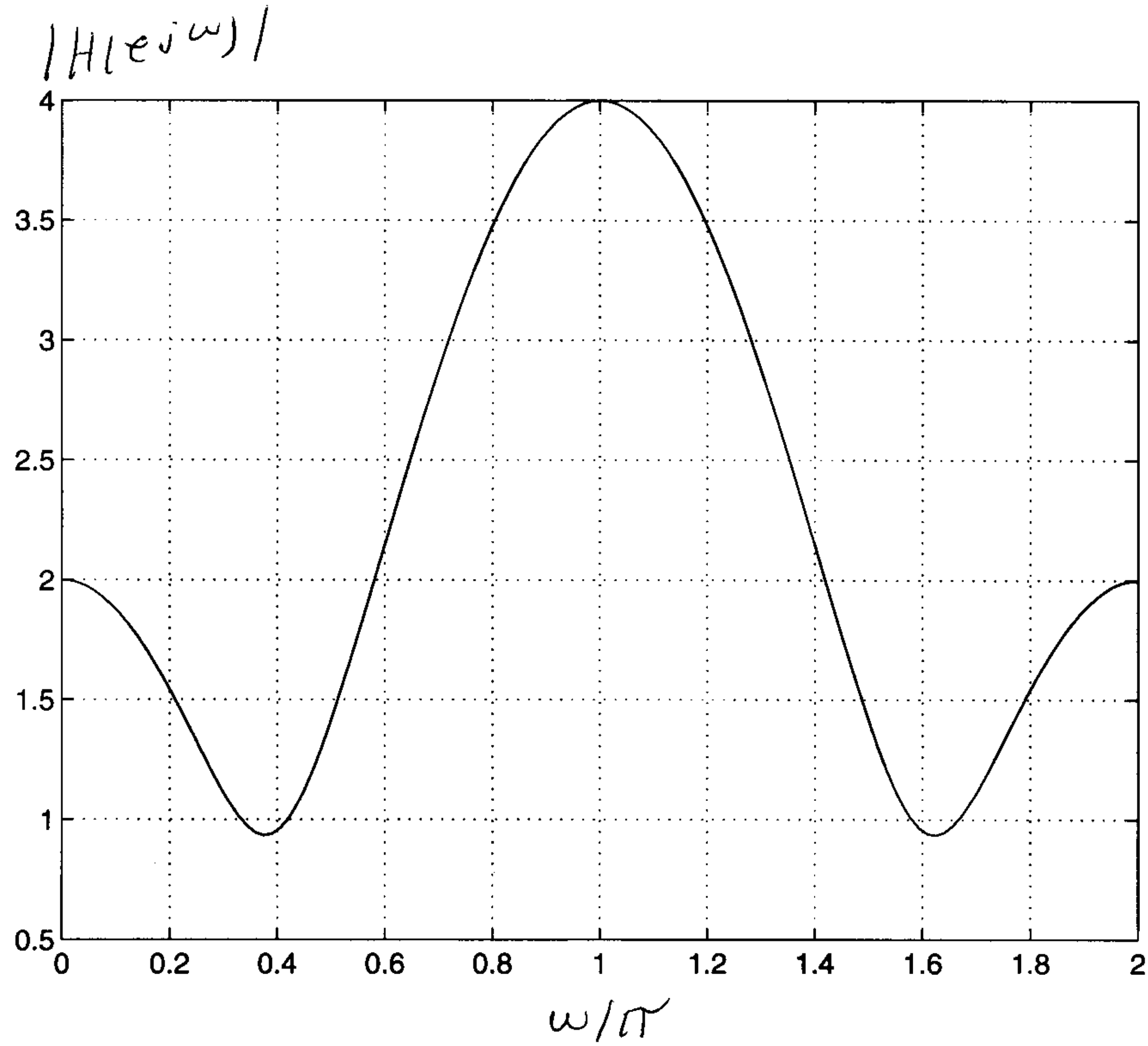
$H(z)$: $2-z^{-1}+z^{-2}$

$$4+6z^{-1}+10z^{-2}+2z^{-3}+12z^{-4}+6z^{-5}$$
$$-2z^{-1}-3z^{-2}-5z^{-3}-z^{-4}-6z^{-5}-3z^{-6}$$
$$4+2z^{-2}+3z^{-3}+5z^{-4}+z^{-5}+6z^{-6}+3z$$

$$Y(z) = 4+4z^{-1}+9z^{-2}+0z^{-3}+16z^{-4}+z^{-5}+3z^{-6}+3z^{-7}$$

$\xrightarrow{z^{-1}}$ $\{4, 4, 9, 0, 16, 1, 3, 3\} = y[n]$

$\uparrow$
$n=0$


|H(e^jω)|
4
3.5
3
2.5
2
1.5
1
0.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
ω/π

4. $H(z) = \dfrac{1}{1-0.9z^{-1}} \Rightarrow y[n] = 0.9y[n-1] + x[n]$

if $x[n] = \delta[n]$: (ASSUME $y[n]=0,\ n<0$)

$y[0] = \delta[0] = 1$
$y[1] = 0.9y[0] + \delta[1] = 0.9$
$y[2] = 0.9y[1] + \delta[2] = 0.9^2$
$\vdots$
$y[n] = 0.9^n = h[n]$

BY LONG DIVISION:

$$
\begin{array}{r l}
 & 1 + 0.9z^{-1} + 0.9^2 z^{-2} + \cdots = H(z) \\
1-0.9z^{-1} \,\big)\!\! & 1 \\
 & \underline{1 - 0.9z^{-1}} \\
 & \quad 0.9z^{-1} \\
 & \quad \underline{0.9z^{-1} - 0.9^2 z^{-2}} \\
 & \qquad\qquad 0.9^2 z^{-2}
\end{array}
$$

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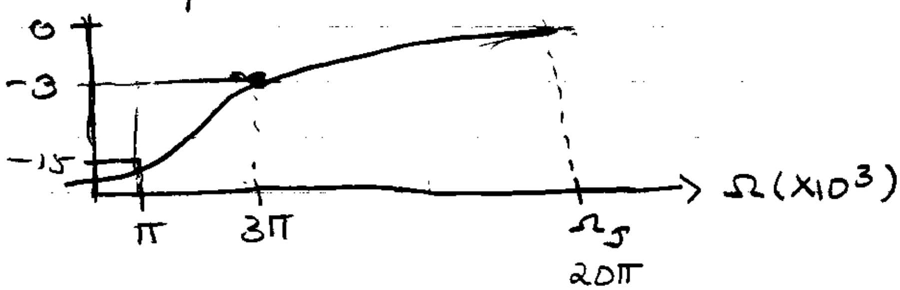
5a) HIGHPASS FILTER DESIGN VIA IMPULSE INVARIANCE

ex) SPECS:
- BUTTERWORTH
- $T = 10^{-4}$
- $|H(j\Omega)| = -3$ dB AT $\Omega_c = 3{,}000\pi$ rad/s
- $|H(j\Omega)| = -15$ dB AT $\Omega \le \Omega_a = 1{,}000\pi \frac{\text{rad}}{s}$

$$\Rightarrow \quad \Omega_s = \frac{2\pi}{T} = 20{,}000\pi \ \text{rad/s}$$



CORRESPONDING $H_{LP}^{p}(j\Omega)$;

Ω_{HP}	$\Omega_{LP}^{p} = \frac{\Omega_c}{\Omega_{HP}}$
0	∞
∞	0
$3{,}000\pi = \Omega_c$	1
$1{,}000\pi = \Omega_a$	$3 = \Omega_a^{p}$

$$N = \frac{\log_{10}(10^{1.5}-1)}{2\log_{10}3} = 1.56 \rightarrow 2$$

$$\therefore H_{LP}^{p}(s) = \frac{1}{s^2+\sqrt{2}\,s+1} = \frac{1}{B_2(s)}$$

$$H_{HP}(s) = H_{LP}^{p}\left(\frac{\Omega_c}{s}\right) = \frac{1}{\left(\frac{\Omega_c}{s}\right)^2+\sqrt{2}\left(\frac{\Omega_c}{s}\right)+1}$$

$$= \frac{s^2}{s^2+\sqrt{2}\,\Omega_c s+\Omega_c^2}$$

AFTER LONG DIVISION

$$H_{HP}(s) = 1 - \frac{\sqrt{2}\,\Omega_c s + \Omega_c^2}{s^2 + \sqrt{2}\,\Omega_c s + \Omega_c^2}$$

$$= 1 - \sqrt{2}\,\Omega_c \frac{(s + \Omega_c/\sqrt{2})}{s^2 + \sqrt{2}\,\Omega_c s + \Omega_c^2}$$

$$= 1 - \sqrt{2}\,\Omega_c \frac{(s + \overbrace{\Omega_c/\sqrt{2}}^{a})}{(s + \Omega_c/\sqrt{2})^2 + \underbrace{\frac{\Omega_c^2}{2}}_{b^2}}$$

AFTER COMPLETING SQUARE

NOW TAKE $\mathcal{L}^{-1}$: $a = b = \Omega_c/\sqrt{2}$

$$h(t) = \delta(t) - \sqrt{2}\,\Omega_c e^{-at} \cos bt \; u(t)$$

$$h[n] = T\,h(t)$$

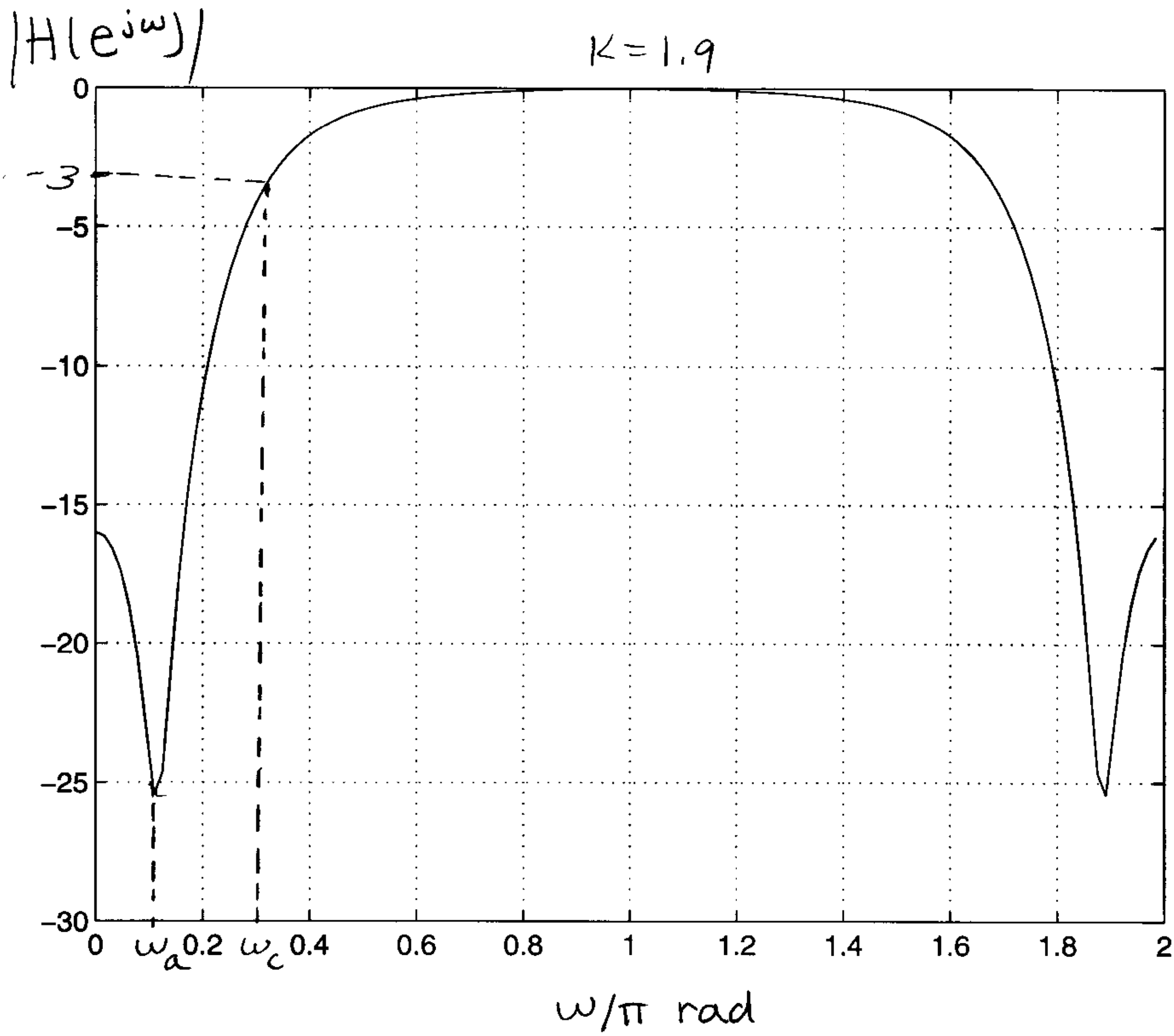
$$= \underbrace{T\delta(0)}_{K}\delta[0] - \sqrt{2}\,\Omega_c T e^{-aTn} \cos bTn \; u[n]$$

$$\Rightarrow H(z) = K - \sqrt{2}\,\Omega_c T \left[\frac{1 - (e^{-a}\cos b) z^{-1}}{1 - (2e^{-a}\cos b) z^{-1} + e^{-2a} z^{-2}} \right]$$

$\delta(0) = \infty$

$K = T\delta(0) = ?$ TRY SEVERAL VALUES UNTIL GET DESIRED HIGHPASS RESPONSE

$K = 1.9$ GIVES GOOD RESULTS



|H(e^{jω})|
K=1.9
0
-3
-5
-10
-15
-20
-25
-30
0
ω_a
0.2
ω_c
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
ω/π rad

5b) BILINEAR TRANSFORM:

DISCRETE TIME SPEC. FREQUENCIES:

$$\omega_c = \Omega_c T = 0.3\pi$$
$$\omega_a = \Omega_a T = 0.1\pi$$

PREWARPED CONT-TIME FREQUENCIES

$$\Omega_c' = \frac{2}{T}\tan(\omega_c/2) = \frac{2}{T}\,0.5095$$
$$\Omega_a' = \frac{2}{T}\tan(\omega_a/2) = \frac{2}{T}\,0.1584$$

NOW DESIGN ANALOG FILTER:

Ω_{HP}	$\Omega_{LP}^{P} = \frac{\Omega_c}{\Omega_{HP}}$
0	∞
∞	0
Ω_c'	1
Ω_a'	$\frac{0.5095}{0.1584} = 3.216$

$M = -15$

$$N = \frac{\log(10^{1.5}-1)}{2\log(3.216)} = 1.465 \rightarrow 2$$

$$H_{LP}^{P}(s) = \frac{1}{s^2 + \sqrt{2}\,s + 1}$$

$$H_{HP}(s) = H_{LP}^{P}\left(\frac{\Omega_c'}{s}\right) = \frac{1}{\left(\frac{\Omega_c'}{s}\right)^2 + \sqrt{2}\left(\frac{\Omega_c'}{s}\right) + 1}$$

$$= \frac{s^2}{s^2 + \Omega_c'\sqrt{2}\,s + (\Omega_c')^2}$$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{1}{\left(\dfrac{\Omega_c' T}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}\right)^2 + \sqrt{2}\left(\dfrac{\Omega_c' T}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}\right) + 1}$$

$$= \frac{(1-z^{-1})^2}{\left[\Omega_c' \frac{T}{2}(1+z^{-1})\right]^2 + \sqrt{2}\,\frac{\Omega_c' T}{2}(1+z^{-1})(1-z^{-1}) + (1-z^{-1})^2}$$

$\Rightarrow$

$$\begin{array}{r} 1 - z^{-1} \\ \times\, 1 - z^{-1} \\ \hline 1 - z^{-1} \\ -z^{-1} + z^{-2} \\ \hline 1 - 2z^{-1} + z^{-2} \end{array}$$

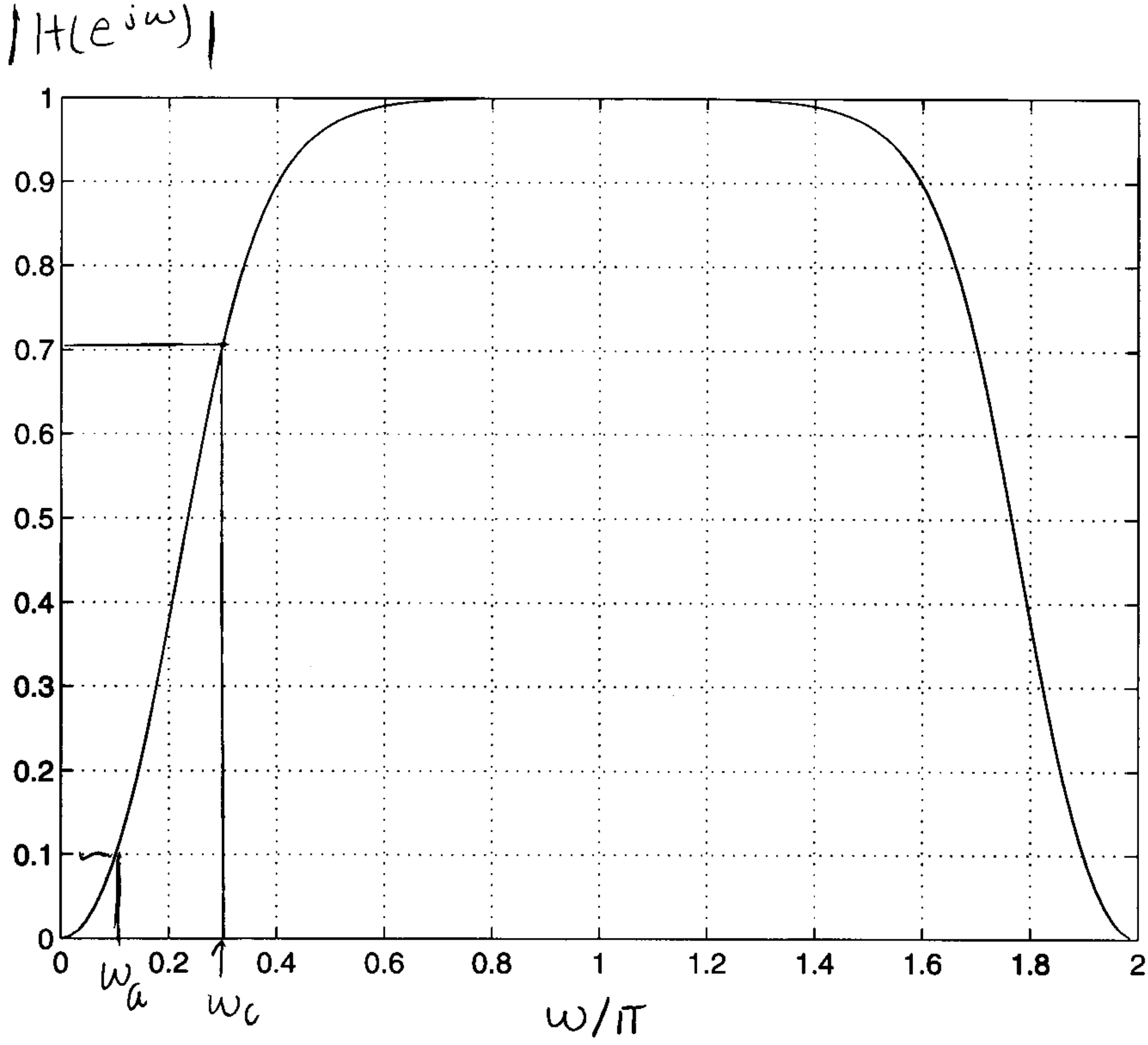
$(1+z^{-1})^2 = 1 + 2z^{-1} + z^{-2}$

$(1+z^{-1})(1-z^{-1}) = 1 - z^{-2}$

$$H(z) = \frac{1-2z^{-1}+z^{-2}}{\left(\frac{\Omega_c' T}{2}\right)^2(1+2z^{-1}+z^{-2}) + \sqrt{2}\,\Omega_c' T/2\,(1-z^{-2}) + 1 - 2z^{-1} + z^{-2}}$$

$$= \frac{1-2z^{-1}+z^{-2}}{\left\{\underbrace{\left[\left(\frac{\Omega_c' T}{2}\right)^2 + \sqrt{2}\,\Omega_c' T/2 + 1\right]}_{1.9801} + \underbrace{\left[2\left(\frac{\Omega_c' T}{2}\right)^2 - 2\right]}_{-1.4808} z^{-1} + \underbrace{\left[\left(\frac{\Omega_c' T}{2}\right)^2 - \frac{\sqrt{2}}{2}\Omega_c' T + 1\right]}_{0.5390} z^{-2}\right\}}$$

$$= \frac{1-2z^{-1}+z^{-2}}{1.9801 - 1.4808\,z^{-1} + 0.5390}$$



$20\log_{10}(0.1) = -20\,dB < -15\ dB$