EE 5345
Biomedical Instrumentation
Lecture 2: slides 37-64

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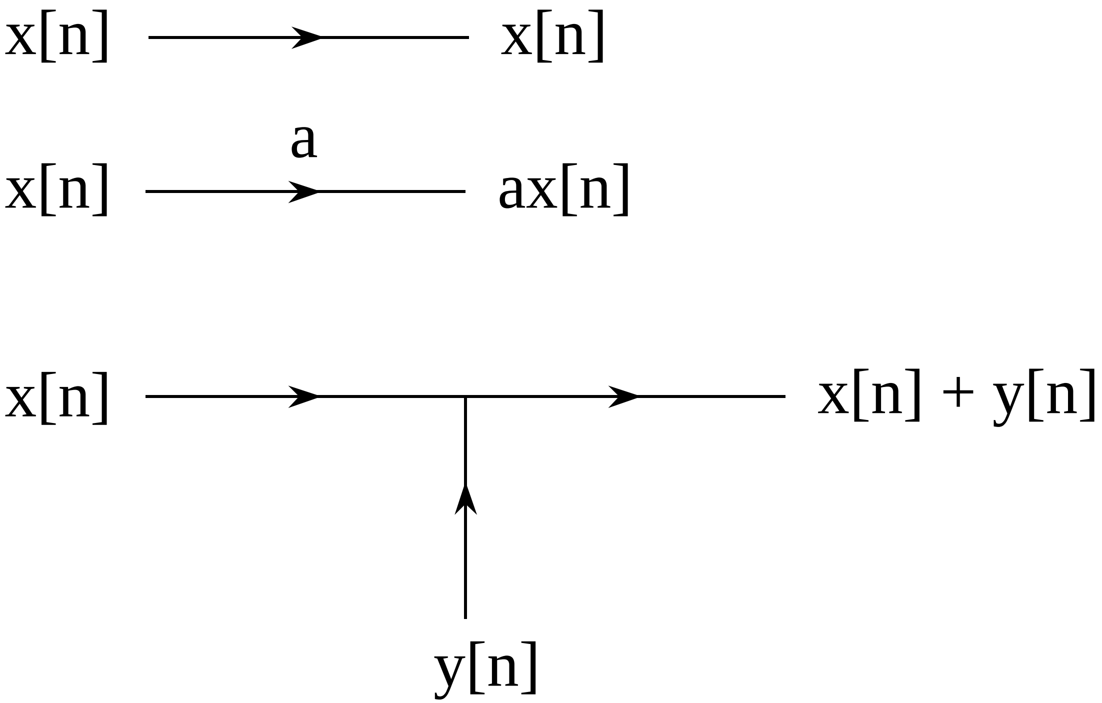
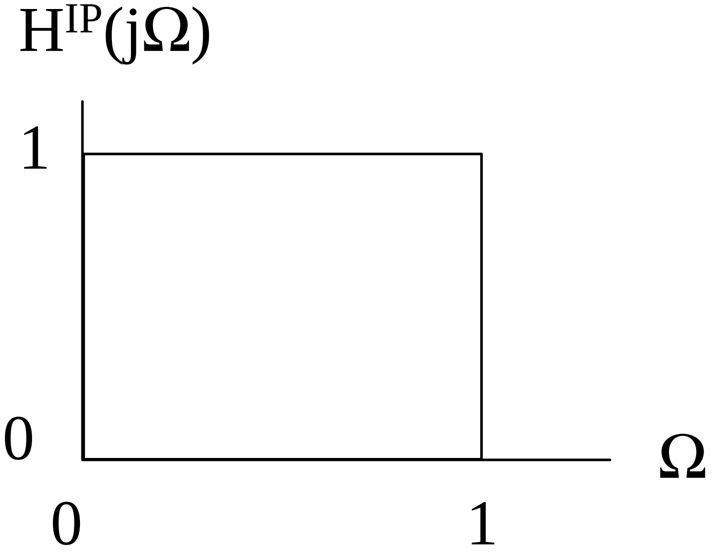
slides can be viewed at:
http:// www.seas.smu.edu/~cd/ee5340.html

Basics of signal flow graphs:



Analog Filter Design

Ideal prototype lowpass filter:



In practice such a filter is impossible to realize, we seek reasonable approximations of this filter.

Nth-order Butterworth Prototype Lowpass Filter:

$$\left|H^{P}(j\Omega)\right|^{2} = \frac{1}{1+\Omega^{2N}}$$

Properties:

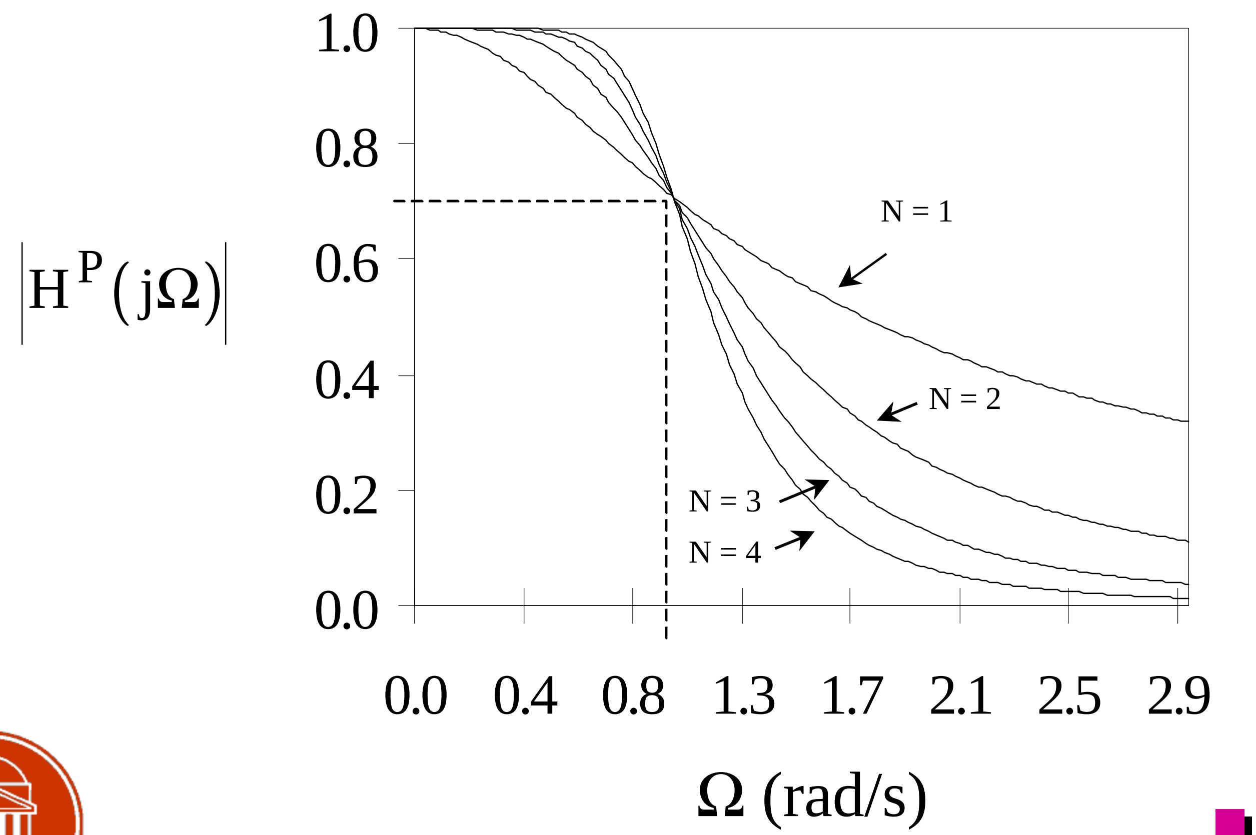
$$\left|H^{P}(j1)\right| = \frac{1}{\sqrt{2}}, \quad \text{all N}$$

$$\left|H^{P}(j\Omega)\right| \to 0, \quad \Omega \to \infty$$

$$\left|H^{P}(j0)\right| = 1, \quad \text{all N}$$

- $\frac{d^{n}}{d\Omega^{n}}\left|H^{P}(j\Omega)\right|_{\Omega=0} = 0$ “maximally flat”

- $H^P(j\Omega)$ is a monotonically decreasing function of Ω.
- Steepness of $H^P(j\Omega)$ is proportional to N:



$$\left|H^{P}(j\Omega)\right|^{2}=\frac{1}{1+\Omega^{2N}}\Rightarrow H^{P}(s)=\frac{1}{B_{N}(s)}$$

$B_N(s)$ = Nth-order Butterworth Polynomial

N	$B_N(s)$
1	s + 1
2	$s^2+\sqrt{2}s+1$

Butterworth Filter Design

Design specification usually given as:

For a cutoff frequency of $\Omega = \Omega_c$, find the correct filter order N so that:

$$\left|H(j\Omega)\right| = M \text{ (dB) at } \Omega = \Omega_a$$

where

$$H(j\Omega) = H^P\left(j\frac{\Omega}{\Omega_c}\right)$$

If the frequency axis is scaled so that Ω_c is mapped to 1 and Ω_a is mapped to $\Omega_a^P = \Omega_a / \Omega_c$, we can work with the prototype filter:

$$M = 20\log_{10}\left|H^P(j\Omega)\right| = 20\log_{10}\left(\frac{1}{\sqrt{1+\left(\Omega_a^P\right)^{2N}}}\right)$$

$$= -10\log_{10}\left(1+\left(\Omega_a^P\right)^{2N}\right)$$

solving for N:

$$N = \frac{\log_{10}\left(10^{-M/10} - 1\right)}{2\log_{10}\left(\Omega_a^P\right)}$$

Butterworth LP Filter Design Procedure:

- Find N for a specified gain M (dB) at $\Omega = \Omega_a$ and cutoff frequency Ω_c using:

$$N = \frac{\log_{10}\left(10^{-M/10} - 1\right)}{2\log_{10}\left(\Omega_a^P\right)}$$

- scale prototype lowpass filter to get desired lowpass filter

$$H(s) = H^P\left(\frac{s}{\Omega_c}\right) = \frac{1}{B_N\left(\frac{s}{\Omega_c}\right)}$$

ex) Design a lowpass Butterworth filter having a cutoff frequency of Ω_c = 10 rad/s with a gain <= -20 dB at Ω_a = 20 rad/s.

using:

$$N = \frac{\log_{10}\left(10^{-M/10} - 1\right)}{2\log_{10}\left(\Omega_a^P\right)}$$

gives:

$$N = \frac{\log_{10}\left(10^{-(-20)/10} - 1\right)}{2\log_{10}\left(\frac{20}{10}\right)} = 3.31$$

must round up to N = 4.

$$H(s) = H^{P}\left(\frac{s}{\Omega_c}\right) = \frac{1}{B_4\left(\frac{s}{\Omega_c}\right)}$$

$B_4(s)$ = 4th order Butterworth polynomial

$B_4(s) = s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s+1$
$B_3(s) = s^3 + 2s^2 + 2s+1$
$B_2(s) = s^2 + 1.4141s + 1$
$B_1(s) = s + 1$

Analog Highpass Filter Design

$$H_{HP}^{P}(j\Omega) = H_{LP}^{P}\left(\frac{1}{j\Omega}\right)$$ is a prototype highpass filter

therefore:

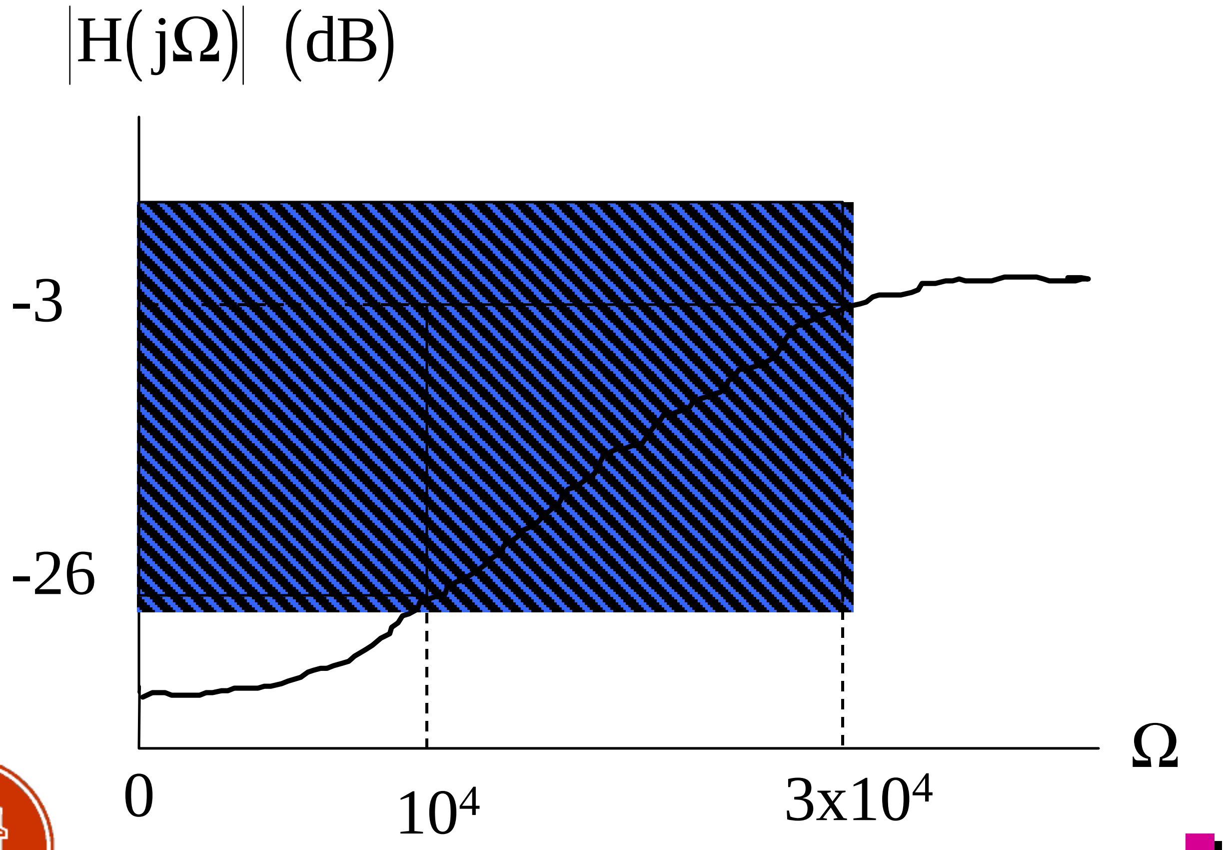
$$H_{HP}(j\Omega) = H_{HP}^{P}\left(j\frac{\Omega}{\Omega_c}\right) = H_{LP}^{P}\left(\frac{\Omega_c}{j\Omega}\right)$$

hence the frequency transformation which converts a prototype lowpass filter to a highpass filter having cutoff frequency Ω_c is:

$$\Omega_{LP}^{P} = \frac{\Omega_c}{\Omega_{HP}}$$

ex) Design a highpass filter having the following specs:

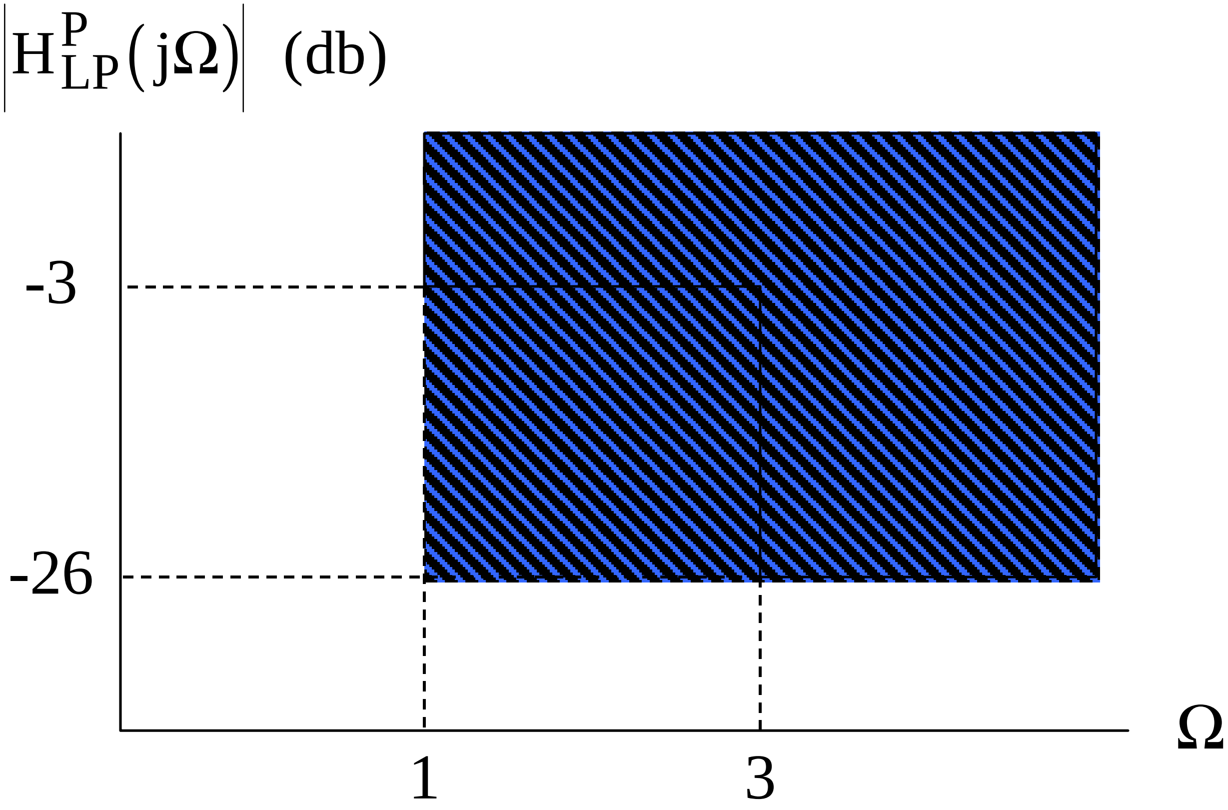
$\Omega_c = 3x10^4$ rad/s
gain < -26 dB, for $\Omega < 10^4$ rad/s



- Find the corresponding LP prototype specs:

Ω_{HP}	$\Omega_{LP}^{P} = \dfrac{\Omega_c}{\Omega_{HP}}$
$\Omega_c = 3\times10^4$	1
$\Omega_a = 10^4$	3
0	∞
∞	0

- Design a prototype lowpass filter $H_{LP}^{P}(s)$ meeting these specs:



$$\Omega_a^P = 3 \ \text{rad/s}$$

- Scale the prototype lowpass filter to get highpass filter:

$$H_{HP}(s) = H_{LP}^{P}\left(\frac{\Omega_c}{s}\right)$$

- Done!

IIR Digital Filter Design by Impulse Invariance

Basic idea:

- one is given design specs for a continuous-time filter and a desired sampling rate.
- design a continuous-time filter, h(t), meeting these specs.
- get h[n] by sampling h(t) as:

$$h[n] = Th(nT)$$

T = sampling rate

- find z-transform of h[n] to get H(z).

ex) Design a lowpass digital filter having a Butterworth characteristic corresponding to the following analog filter:

$$\Omega_c = 2000\pi \text{ rad/s}, \; T = 10^{-4} \text{ s}$$

$$|H(j\Omega)| \leq 0.5, \quad \Omega \geq 4{,}000\pi = \Omega_a \quad \text{rad / s}$$

find desired filter order:

$$N = \frac{\log_{10}\left(10^{-M/10} - 1\right)}{2\log_{10}\left(\Omega_a^P\right)} \qquad \Omega_a^P = \frac{\Omega_a}{\Omega_c} = \frac{4{,}000\pi}{2{,}000\pi} = 2$$

$$M = 20\log_{10}(0.5) = -6 \qquad N = \frac{\log_{10}\left(10^{-(-6)/10} - 1\right)}{2\log_{10}(2)} = 0.788 \rightarrow 1$$

$$\Omega_c = 0.2\pi / T$$

$$H(s) = H^P\left(\frac{s}{\Omega_c}\right) = \frac{1}{B_1\left(\frac{s}{\Omega_c}\right)} = \frac{1}{\frac{s}{\Omega_c}+1} = \frac{1}{\frac{sT}{0.2\pi}+1} = \frac{0.2\pi / T}{s+0.2\pi / T}$$

take inverse Laplace Transform:

$$h(t) = \frac{0.2\pi}{T} e^{-0.2\pi t/T} u(t)$$

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- sample impulse response:

$$h[n] = Th(nT)$$

$$= 0.2\pi e^{-0.2\pi n} u[n] \qquad u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

note that h[n] is independent of T.

- take z-transform of h[n]:

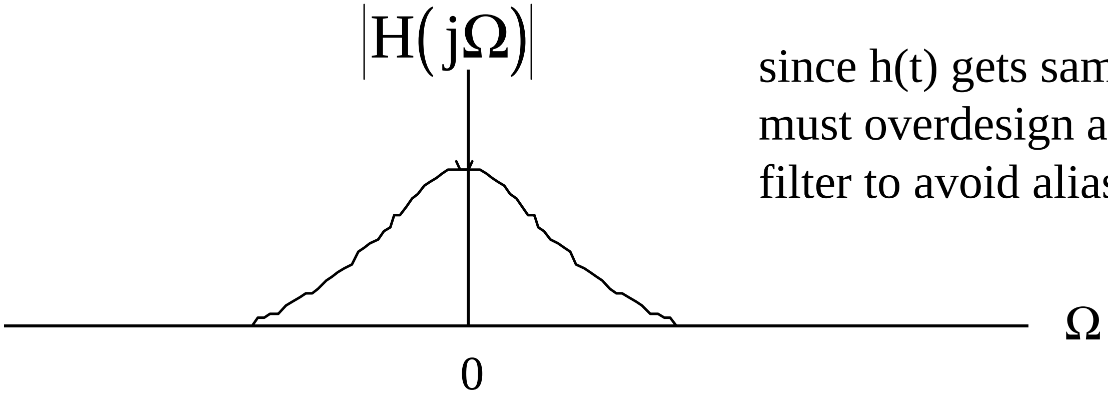
$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n} = 0.2\pi \sum_{n=0}^{\infty} e^{-0.2\pi n} z^{-n}$$

$$= \frac{0.2\pi}{1 - e^{-0.2\pi} z^{-1}} = \frac{Y(z)}{X(z)}$$

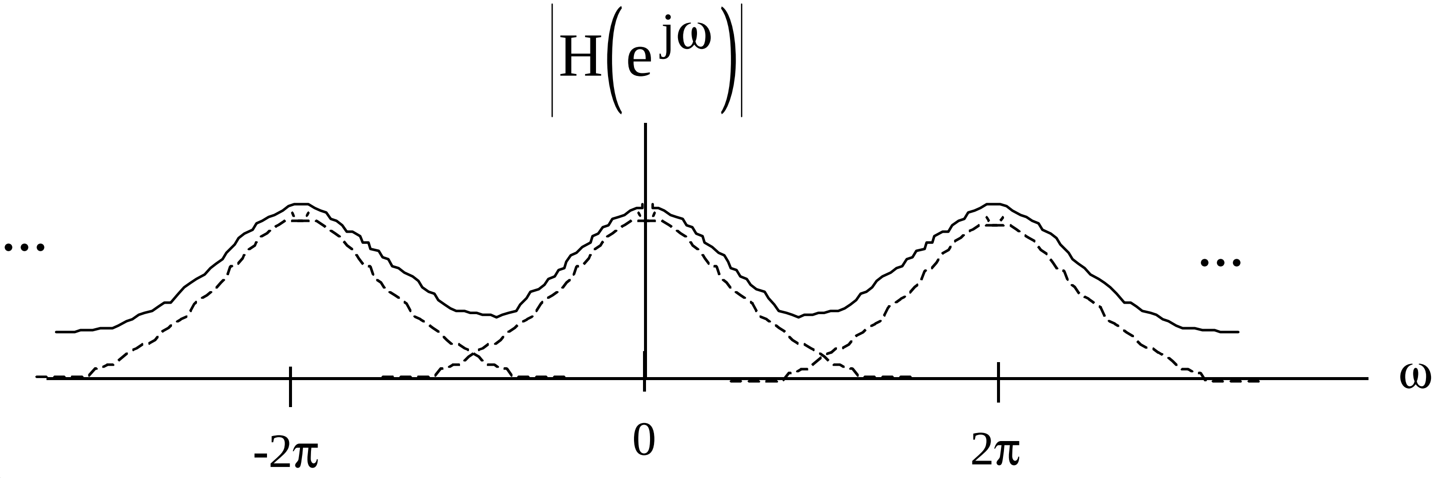
- find digital filter difference equation:

$$y[n] = 0.2\pi x[n] + e^{-0.2\pi} y[n-1]$$

- done!



since h(t) gets sampled, must overdesign analog filter to avoid aliasing.



Some Laplace Transform Pairs

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}$$

$$f(at) \leftrightarrow \frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0$$

$$e^{-at}\cos bt\, u(t) \leftrightarrow \frac{s+a}{(s+a)^2+b^2}$$

$$e^{-at}\sin bt\, u(t) \leftrightarrow \frac{b}{(s+a)^2+b^2}$$

Some Useful Z-Transform Pairs:

$$e^{-an}\cos bn\, u[n] \leftrightarrow \frac{1-\left[e^{-a}\cos b\right]z^{-1}}{1-\left[2e^{-a}\cos b\right]z^{-1}+e^{-2a}z^{-2}}$$

$$e^{-an}\sin bn\, u[n] \leftrightarrow \frac{\left[e^{-a}\sin b\right]z^{-1}}{1-\left[2e^{-a}\cos b\right]z^{-1}+e^{-2a}z^{-2}}$$

ex) Design a lowpass digital filter having a Butterworth characteristic corresponding to the following analog filter:

Ω_c = 2000π rad/s, T = 10^{-4} s

$$\left|H\left(j\Omega\right)\right| \le 0.3, \quad \Omega \ge 4{,}000\pi = \Omega_a \quad \text{rad / s}$$

find desired filter order:

$$N = \frac{\log_{10}\left(10^{-M/10} - 1\right)}{2\log_{10}\left(\Omega_a^P\right)} \qquad \Omega_a^P = \frac{\Omega_a}{\Omega_c} = \frac{4{,}000\pi}{2{,}000\pi} = 2$$

M = 20$\log_{10}$(0.3) = -10.46 $N = \dfrac{\log_{10}\left(10^{-(-10.46)/10} - 1\right)}{2\log_{10}\left(2\right)} = 1.67 \to 2$

$$H(s) = H^{P}\left(\frac{s}{\Omega_c}\right) = \frac{1}{B_2\left(\frac{s}{\Omega_c}\right)}$$

$$= \frac{1}{\left(\frac{s}{\Omega_c}\right)^2 + \sqrt{2}\,\frac{s}{\Omega_c} + 1}$$

$$= \frac{\Omega_c^2}{s^2 + \sqrt{2}\Omega_c s + \Omega_c^2}$$

complete the square:

take 1/2 the linear term, square it, then add it and subtract it.

$$H(s) = \frac{\overbrace{\frac{\Omega_c}{\sqrt{2}}}^{b}\sqrt{2}\Omega_c}{\left(s+\underbrace{\frac{\Omega_c}{\sqrt{2}}}_{a}\right)^2+\overbrace{\frac{\Omega_c^2}{2}}^{b^2}}$$

inverse Laplace Transform:

$$h(t) = \sqrt{2}\Omega_c e^{-at}\sin bt\, u(t)$$

sample h(t):

$$h[n] = Th(nT)$$

$$= \sqrt{2}\Omega_c T e^{-anT} \sin bnT\ u[n]$$

from z-transform table:

$$H(z) = \frac{\left[e^{-aT} \sin bT\right] z^{-1}}{1 - \left[2e^{-aT} \cos bT\right] z^{-1} + e^{-2aT} z^{-2}}$$

$$\Omega_c = 0.2\pi / T \qquad aT = bT = \frac{\Omega_c}{\sqrt{2}} T = \frac{0.2\pi}{\sqrt{2}}$$

difference equation:

$$y[n] = \left[e^{-aT}\sin bT\right] x[n-1] + \left[2e^{-aT}\cos bT\right] y[n-1] - e^{-2aT} y[n-2]$$