EDICS category 5.2.1

An Algorithm for Pole-Zero System Model Order Estimation

Carlos E. Davila and Hsia-Ling Chiang

Electrical Engineering Department, Southern Methodist University
Dallas, Texas 75275
cd@seas.smu.edu

Abstract

In system identification, estimates of the unknown system model orders are often required. An algorithm for estimating model orders is described which looks at input/output data covariance matrix eigenvectors. When model orders are overestimated, zeros appear in the noise subspace eigenvectors. The number of zeros present can be used to estimate model orders.

October 20, 1994

I. Introduction

The identification of pole-zero systems using the eigen- or singular value decomposition of the input/output data covariance matrix has been known for some time [1, 2, 3, 4, 5]. These techniques offer the advantage of giving unbiased parameter estimates when both the input and output data are subject to error. More recently, there has been a renewed interest in Total Least Squares (TLS) for parameter identification and frequency estimation [6, 7, 8, 9] Several variants of TLS have been described in [10, 11, 12, 13]. Total Least Squares can be seen to be closely related to the earlier eigen-decomposition techniques. When model orders are overestimated, the usual approach is to compute the eigenvectors spanning the noise subspace of the input/output data covariance matrix and then find the linear combination of these eigenvectors having a constrained minimum norm [8, 6]. It can be easily shown however that this minimum norm solution will not produce correct parameter estimates. When TLS is applied to pole-zero system identification, it becomes necessary that the model order be known *a priori*. It is shown in [8] that for the overestimated model order case, the noise subspace eigenvectors can be used to estimate the frequency response of the unknown system; however, some applications like model reference adaptive control, require explicit knowledge of the unknown system parameters [14].

This paper describes a method of estimating the model order of a pole-zero system using the noise subspace eigenvectors of the sample data covariance matrix. When model orders are overestimated, zeros appear in the noise subspace eigenvectors. The number of zeros can then be used to estimate model order. A related approach was recently described by Liang, Wilkes, and Cadzow which looks at the eigenvalues of the data covariance matrix [15]. The method in [15] requires the evaluation of covariance matrix eigenvalues corresponding to numerous possible model orders before a final choice can be made. Eigenvector entries appear to undergo more abrupt transitions from nonzero values to small (ideally zero) values. Eigenvalues, on the other hand, may undergo small transitions in amplitude in going from the signal to the noise subspace, thereby making it difficult to determine the noise subspace dimension.

II. Background

Consider the pole-zero system having no pole-zero cancellations satisfying

$$y(n) = \sum_{i=1}^{N-1} a_i(n)y(n-i) + \sum_{j=0}^{M-1} b_j(n)x(n-j) \tag{1}$$

When prewindowed, noisy measurements, $\tilde{y}(n)$ and $\tilde{x}(n)$, of the input and output are available up to time n (see Fig. 1), the TLS method computes the $(N+M)\times(N+M)$ covariance matrix

$$\overline{R}_n = \sum_{i=0}^{n} \overline{\phi}_i \overline{\phi}_i^T \tag{2}$$

where,

$$\overline{\phi}_n = [\tilde{y}(n)\tilde{y}(n-1)\cdots\tilde{y}(n-N+1)\tilde{x}(n)\tilde{x}(n-1)\cdots\tilde{x}(n-M)]^T$$

$\tilde{y}(n) = y(n) + v_o(n)$, and $\tilde{x}(n) = x(n) + v_i(n)$. Here, $v_o(n)$ and $v_i(n)$ are uncorrelated, zero-mean white measurement noise processes having variance σ_o^2 and σ_i^2, respectively. If the MA and AR model orders have been estimated correctly, the parameter estimates are obtained from the generalized eigenvector associated with the minimum generalized eigenvalue satisfying $\overline{R}q = \lambda\overline{D}q$, with [1]

$$\overline{D} = \begin{bmatrix} \frac{\sigma_o^2}{\sigma_i^2} I_N & 0 \\ 0 & I_M \end{bmatrix} \tag{3}$$

Only the ratio of output to input measurement noise variance is required. If the input $x(n)$ is assumed known with zero error, the matrix $\overline{D}$ is

$$\overline{D} = \begin{bmatrix} I_N & 0 \\ 0 & 0_M \end{bmatrix} \tag{4}$$

The parameter estimates are then obtained as

$$\hat{\theta} \equiv \left[\hat{a}_1 \cdots \hat{a}_{N-1} \hat{b}_0 \cdots \hat{b}_{M-1}\right]^T = \frac{-[q]_{2,N+M}}{[q]_1} \tag{5}$$

where $[q]_1$ and $[q]_{2,N+M}$ are the first and second through $(N+M)$th entries of q, respectively. Many analytic results on TLS and its variants along with additional references appear in [16]. The existence and uniqueness of the TLS solution for pole-zero system identification has been considered in [8, 16, 17]. It is shown in [17] that a unique TLS solution is possible if either the MA or AR model order is overestimated. However, if both the AR and MA model orders are overestimated, the TLS solution will not be unique. The proof of these assertions results from examining the nullspace of the noiseless ($v_i(n) = v_o(n) = 0$) covariance matrix $\overline{R}_n$. This nullspace is spanned by the generalized eigenvector(s) corresponding to the minimum generalized eigenvalue of the noisy version of $\overline{R}_n$. Consequently, examination of the dimension of this nullspace makes it possible to determine the uniqueness of the TLS solution: a one dimensional nullspace giving a unique solution. The noiseless data vector can be expressed as [18]

$$\overline{\phi}_n = \begin{bmatrix} y(n) \\ y(n-1) \\ \vdots \\ y(n-N+1) \\ x(n) \\ \vdots \\ x(n-M+1) \end{bmatrix} = \mathcal{L}(B,A) \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-N-M+2) \end{bmatrix} \frac{1}{A\left(\Delta^{-1}\right)} \tag{6}$$

where $A(z) = 1 - \sum_{i=1}^{N-1} a_i z^i$, Δ^{-1} is the unit delay operator and

$$\mathcal{L}(B,A) = \left[\begin{array}{ccccccc} b_0 & b_1 & \cdots & b_{M-1} & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_{M-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_{M-1} \\ 1 & -a_1 & \cdots & -a_{N-1} & 0 & \cdots & 0 \\ 0 & 1 & -a_1 & \cdots & -a_{N'-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -a_1 & \cdots & -a_{N-1} \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} N \\ \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} M \end{array} \tag{7}$$

is the $(N+M) \times (N+M-1)$ Sylvester matrix. The data covariance matrix for this data vector can then be expressed as

$$\overline{R}_n = \mathcal{L}(B,A) \sum_{i=0}^{n} \tilde{\phi}_n \tilde{\phi}_n^T \mathcal{L}(B,A)^T \tag{8}$$

with

$$\tilde{\phi}_n = \left[\begin{array}{c} x(n) \\ \vdots \\ x(n-M-M+2) \end{array}\right] \frac{1}{A\left(\Delta^{-1}\right)} \tag{9}$$

The inner summation in (8) is positive definite provided $x(n)$ is persistently exciting of at least order $N+M-1$ [18]. Correspondingly, the nullspace of $\overline{R}_n$ can be examined by looking at the left nullspace of $\mathcal{L}(B,A)$.

Theorem 1: The left nullspace of (7) is spanned by

$$\left[\begin{array}{ccccccc} 1 & -a_1 & \cdots & -a_{N-1} & -b_0 & \cdots & -b_{M-1} \end{array}\right] \tag{10}$$

Proof: Multiplying by $\mathcal{L}(B,A)$, it is easy to verify that its left nullspace contains (10). The dimension of the left nullspace of $\mathcal{L}(B,A)$ is the difference between the number of rows of $\mathcal{L}(B,A)$, $N+M$, and the rank of $\mathcal{L}(B,A)$, $N+M-1$ ([18], Appendix A.6), [19]. □

Now suppose the MA and AR model orders have been overestimated as $M' > M$ and $N' > N$, respectively. The following $(N' + M') \times max(M + N' - 1, N + M' - 1)$ Sylvester matrix results:

$$\overline{\mathcal{L}}(B,A) = \left[\begin{array}{cccccccc} b_0 & b_1 & \cdots & b_{M-1} & 0 & \cdots & 0 & (0) \\ 0 & b_0 & b_1 & \cdots & b_{M-1} & \cdots & 0 & (0) \\ \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \vdots & (0) \\ 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_{M-1} & (0) \\ 1 & -a_1 & \cdots & -a_{N-1} & 0 & \cdots & 0 & (0) \\ 0 & 1 & -a_1 & \cdots & -a_{N-1} & \cdots & 0 & (0) \\ \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \vdots & (0) \\ 0 & \cdots & 0 & 1 & -a_1 & \cdots & -a_{N-1} & (0) \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} N' \\ \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} M' \end{array} \tag{11}$$

The notation "(0)" denotes the possibility of one or more zeros depending on whether $M' - M > N' - N, M' - M < N' - N$, or $M' - M = N' - N$.

Theorem 2: The left nullspace of $\overline{\mathcal{L}}(B,A)$ is spanned by the $min(M' - M, N' - N) + 1, (N' + M') \times 1$ vectors:

$$
\begin{bmatrix} 1 \\ -a_1 \\ \vdots \\ -a_{N-1} \\ 0 \\ \vdots \\ 0 \\ -b_0 \\ \vdots \\ -b_{M-1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}}
\begin{matrix} \\ \\ \\ \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} N'-N \\ \\ \\ \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} M'-M \\ \\ \end{matrix}
\quad
\begin{bmatrix} 0 \\ 1 \\ -a_1 \\ \vdots \\ -a_{N-1} \\ 0 \\ \vdots \\ 0 \\ -b_0 \\ \vdots \\ -b_{M-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}^{\mathrm{T}}
\begin{matrix} \\ \\ \\ \\ \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} N'-N \\ \\ \\ \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} M'-M-1 \\ \\ \end{matrix}
\quad \cdots \quad
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -a_1 \\ \vdots \\ -a_{N-1} \\ 0 \\ \vdots \\ 0 \\ -b_0 \\ \vdots \\ -b_{M-1} \\ (0) \end{bmatrix}^{\mathrm{T}}
\begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} M'-M-\gamma \\ \\ \\ \\ \\ \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} N'-N \\ \\ \\ \\ \left.\begin{matrix} \\ \end{matrix}\right\} \gamma \end{matrix}
\tag{12}
$$

where the parentheses indicate the number of consecutive zeros and $\gamma = max(0, M' - M - (N' - N))$. Moreover, these vectors form a basis for the left nullspace of $\overline{\mathcal{L}}(B, A)$.

Proof: Each of the above linearly independent vectors can be verified to be in the left nullspace of $\overline{\mathcal{L}}(B, A)$ by directly multiplying out. Subtracting the rank of $\overline{\mathcal{L}}(B, A)$, $max(M + N' - 1, N + M' - 1)$ (see [18]), from the number of rows of $\overline{\mathcal{L}}(B, A)$, the left nullspace dimension of $\overline{\mathcal{L}}(B, A)$ is seen to be $min(M' - M, N' - N) + 1$. A similar result is found in [8]. □

The left nullspace of $\overline{\mathcal{L}}(B, A)$, or equivalently the nullspace of $\overline{R}_n$ is usually referred to as the "noise subspace". An immediate consequence of *Theorem 2* is:

Theorem 3:

- When $M' - M > N' - N$, the last $(M' - M) - (N' - N)$ entries of all noise subspace eigenvectors are zero.
- When $M' - M < N' - N$, there are $(N' - N) - (M' - M)$ zero entries after the first $M' - M + N$ (usually nonzero) entries of the noise subspace eigenvectors.

- When $M' - M = N' - N$, there are no guaranteed zero entries in the noise subspace eigenvectors.

Theorem 3 is illustrated via several examples in the Appendix. By counting the number of zero entries within the noise subspace eigenvectors, it becomes possible to estimate N and M.

III. Model Order Estimation Algorithm

1. It is assumed that M' and N' have been chosen so that the noise subspace has a dimension of one or greater. This can be accomplished by checking to see that there are a sufficient number of small eigenvalues of $\overline{R}_n$ which can be assumed to be associated with a noise subspace. This may require several eigendecompositions.

2. Compute the eigendecomposition of the $(M' + N') \times (M' + N')$ covariance matrix $\overline{R}_n$. Assuming no noise is present in the input measurements, the eigenvalue problem to be solved is $\overline{R}_n q = \lambda \overline{D} q$, where $\overline{D}$ is defined in (4). The output measurements should undergo noise reduction as described in [15].

3. Choose a candidate pair of model order estimates $\hat{M}$ and $\hat{N}$.

4. Determine the location and size of the zero block which would occur if $\hat{M} = M$ and $\hat{N} = N$. The number of columns in the zero block is the noise subspace dimension and can be determined as $min(M' - \hat{M}, N' - \hat{N}) + 1$. The number of rows in the zero block is given by $|(N' - \hat{N}) - (M' - \hat{M})|$. If $N' - \hat{N} > M' - \hat{M}$, then the zero block starts at row $M' - \hat{M} + \hat{N} + 1$ of the noise subspace eigenvectors. If $N' - \hat{N} < M' - \hat{M}$, it is located at the "bottom" of the noise subspace eigenvectors.

5. Compute the average power, $\sigma_{\hat{N},\hat{M}}$, of the assumed zero block entries and enter the result in a table indexed by $\hat{M}$ and $\hat{N}$.

6. The correct model order is the "small" entry in the $\sigma_{\hat{N},\hat{M}}$ table that is closest to the (1,1) entry. This can be found by taking the ratio of adjacent $\sigma_{\hat{N},\hat{M}}$ table entries along all columns and along all rows, generating two additional tables or matrices of ratios, indexed by $\hat{M}$ and $\hat{N}$. The Hadamard (element by element) product of these two matrices is then searched for the largest entry, which corresponds to the model order estimate. For $\hat{M} < M$ or $\hat{N} < N$, the assumed zero block will not lie strictly within the actual zero block, hence the entries in the $\sigma_{\hat{N},\hat{M}}$ table for these values of $\hat{M}$ and $\hat{N}$ will tend to be larger than those entries for which the assumed zero block lies entirely within the actual zero block.

To reduce the SNR at which the proposed method yields correct model order estimates, the $\sigma_{\hat{N},\hat{M}}$ table can be averaged over different values of M' and N' (i.e. different eigendecompositions). This has the effect of reducing the effect of outliers. Moreover, by fixing M' and varying N' or vice versa, the possibility of having $M' - M = N' - N$ for each eigendecomposition is averted.

IV. Experimental Results

Experiments similar to those described in [15] were performed.

Experiment 1: The unknown system satisfied the difference equation

$$\begin{aligned} y(n) \;=\; & 1.2798y(n-1) - 0.7805y(n-2) + 0.1635y(n-3) \\ & -0.7566y(n-4) + 1.0621y(n-5) - 0.7821y(n-6) \\ & +x(n) - 0.2997x(n-1) + 0.4147x(n-2) \\ & -0.2794x(n-3) + 0.4973x(n-4) \end{aligned} \tag{13}$$

giving $N = 7$ and $M = 5$. Noise was added to the unknown system output to simulate measurement noise giving an SNR of 10 dB. The unknown system input and output was measured for 1,500 time samples. The noise reduction technique described in [15] was applied to increase the SNR of the output measurements. Figure 2 shows the noise subspace

eigenvector entries for $M' = 12$ and $N' = 10$. The square in the lower right surrounds the entries which are supposed to be zero, in theory. They are not zero due to perturbations in the eigenvector estimates from the noise in the data and finite sample size. Fig. 3 shows the $\sigma_{\hat{N},\hat{M}}$ table for the eigenvectors in Fig. 2. The region marked off within the $\sigma_{\hat{N},\hat{M}}$ table corresponds to assumed zero blocks which lie entirely within the actual zero block. Figure 4 shows the Hadamard product of the row and column ratios of the $\sigma_{\hat{N},\hat{M}}$ table. The largest entry corresponds to the true model order.

To study the performance of the method at different SNR's, a series of experiments were performed at SNR's ranging from 2 dB to 20 dB in 2 dB steps. For each SNR, overestimated model orders were set to $N' = 10$ while M' was varied from 10 to 19. This resulted in 10 different eigendecompositions. For each eigendecomposition a $\sigma_{\hat{N},\hat{M}}$ table was generated. The 10 tables were subsequently averaged and a model order estimate was obtained from the average $\sigma_{\hat{N},\hat{M}}$ table. For each SNR, 25 independent trials were run and the percent correct model order estimation was computed. The proposed eigenvector method (EVEC) was compared with the eigenvalue-based method (EVAL) of [15]. For the eigenvalue method, the same generalized eigendecomposition used for the eigenvector method was used. The generalized eigendecomposition appears to give better results than the conventional eigendecomposition since the former does not account for differences in input and output noise levels. The results of the comparison are illustrated in Fig. 5. The proposed method is seen to give correct model order estimates at lower SNR values that the eigenvalue method.

Experiment 2: The above experiment was repeated, with the unknown system satisfying

$$\begin{aligned} y(n) &= -0.7907y(n-1) - 0.042y(n-2) + 0.5556y(n-3) \\ &\quad +0.0247y(n-4) - 0.3846y(n-5) - 0.3026y(n-6)) \\ &\quad +x(n) + 0.3452x(n-1) + 0.53x(n-2) \\ &\quad +0.3985x(n-3) + 0.8138x(n-4) \end{aligned} \tag{14}$$

Here, $N = 7$ and $M = 5$ and the poles have been chosen to be closer to the origin than

in *Experiment 1.* The results are shown in Fig. 6. Again, the proposed method outperforms the eigenvalue method, though the improvement is not as pronounced is in *Experiment 1.*

V. Summary

This paper describes a method of estimating the model orders of pole-zero systems which uses the entries of the noise subspace eigenvectors. The method was found to out-perform a recently published eigenvalue-based method.

VI. Appendix

For each example, the output of the unknown IIR system obeys

$$y(n) = a_1 y(n-1) + b_0 x(n) + b_1 x(n-2) + b_2 x(n-2) \tag{15}$$

so that $M = 3$ (MA order) and $N = 2$ (AR order).

Example 1: Suppose that both the AR and MA orders have been overestimated so that $N' = 3$ and $M' = 5$ so that $(M' - M) > (N' - N)$. The data vector becomes

$$\begin{bmatrix} y(n) \\ y(n-1) \\ y(n-2) \\ x(n) \\ x(n-1) \\ x(n-2) \\ x(n-3) \\ x(n-4) \end{bmatrix} = \begin{bmatrix} b_0 & b_1 & b_2 & 0 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & 0 \\ 1 & -a_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -a_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -a_1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -a_1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ x(n-3) \\ x(n-4) \\ x(n-5) \end{bmatrix} \frac{1}{1 - a_1 \Delta^{-1}} \tag{16}$$

The Sylvester matrix left nullspace is spanned by $\begin{bmatrix} 1 & -a_1 & 0 & -b_0 & -b_1 & -b_2 & 0 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 & -a_1 & 0 & -b_0 & -b_1 & -b_2 & 0 \end{bmatrix}^T$. Since the 2 noise subspace eigenvectors also span this same nullspace, they will also have zeros in their last $(M'-M)-(N'-N) = (5-3)-(3-2) = 1$ entry.

Example 2: Finally suppose that $N'=5$ and $M'=4$ so that $(N'-N) > (M'-M)$. The data vector becomes

$$\begin{bmatrix} y(n) \\ y(n-1) \\ y(n-2) \\ y(n-3) \\ y(n-4) \\ x(n) \\ x(n-1) \\ x(n-2) \\ x(n-3) \end{bmatrix} = \begin{bmatrix} b_0 & b_1 & b_2 & 0 & 0 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 & 0 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & 0 & 0 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & 0 & 0 & b_0 & b_1 & b_2 \\ 1 & -a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -a_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ x(n-3) \\ x(n-4) \\ x(n-5) \\ x(n-6) \end{bmatrix} \frac{1}{1-a_1\Delta^{-1}} \qquad (17)$$

The Sylvester matrix left nullspace is spanned by $\begin{bmatrix} 1 & -a_1 & 0 & 0 & 0 & -b_0 & -b_1 & -b_2 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 & -a_1 & 0 & 0 & 0 & -b_0 & -b_1 & -b_2 \end{bmatrix}^T$. Note that any vector in the noise subspace has $(N'-N)-(M'-M) = 3$ zeros starting with the $M'-M+N+1 = 4$th entry.

References

[1] S. Sagara and K. Wada, "On-line modified least-squares parameter estimation of linear discrete dynamic systems," *Int. J. Control*, vol. 25, pp. 329–343, 1977.

[2] H. Sakai and M. Arase, "Recursive parameter estimation of an autoregressive process disturbed by white noise," *Int. J. Control*, vol. 30, pp. 949–966, 1979.

[3] M. Levin, "Estimation of a system pulse transfer function in the presence of noise," *IEEE Trans. Automatic Control*, vol. AC-9, pp. 229–235, July 1964.

[4] K. Furuta and J. Paquet, "On the identification of time-invariant discrete processes," *IEEE Trans. Automatic Control*, vol. AC-15, pp. 153–155, Feb 1970.

[5] L. Klein, "The estimation of distributed lags," *Econometrica*, vol. 26, pp. 553–565, 1958.

[6] G. Golub and C. V. Loan, "An analysis of the total least squares problem," *SIAM J. Numer. Anal.*, vol. 17, pp. 883–893, Dec. 1980.

[7] S. V. Huffel and J. Vandewalle, "The use of total linear least squares for identification and parameter estimation," in *IFAC Identification and System Parameter Estimation 1985*, (York, U.K.), pp. 1167–1172, 1985.

[8] J. Cadzow and O. Solomon, "Algebraic approach to system identification," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-34, pp. 462–469, June 1986.

[9] M. Rahman and K. Yu, "Total least squares approach for frequency estimation using linear prediction," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 1440–1454, October 1987.

[10] T. Abatzoglou, J. Mendel, and G. Harada, "Constrained total least squares," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 1440–1454, October 1987.

[11] S. V. Huffel and J. Vandewalle, "Analysis and solution of the nongeneric total least squares problem," *SIAM J. Matrix Anal. Appl.*, vol. 9, pp. 360–372, 1988.

[12] S. V. Huffel and H. Zha, "The restricted total least squares problem: Formulation, algorithm and propoerties," *SIAM J. Matrix Anal. Appl.*, vol. 12, pp. 292–309, 1991.

[13] R. DeGroat and E. Dowling, "The data least squares problem and channel equalization," *IEEE Trans. Signal Processing*, vol. SP-41, pp. 407–410, 1993.

[14] Y. Landau, *Adaptive Control*. Dekker, 1979.

[15] G. Liang, D. Wilkes, and J. Cadzow, "ARMA model order estimation based on the eigenvalues of the covariance matrix," *IEEE Trans. Signal Processing*, vol. SP-41, pp. 3003–3009, October 1993.

[16] S. V. Huffel and J. Vandewalle, *The Total Least Squares Problem: Computational aspects and analysis*. SIAM, 1991.

[17] C. Davila, "An algorithm for efficient, unbiased, equation-error infinite impulse response adaptive filtering," *IEEE Trans. Signal Processing*, vol. SP-42, Feb. 1994.

[18] T. Soderstrom and P. Stoica, *System Identification*. Prentice Hall, 1989.

[19] G. Strang, *Linear Algebra and its Applications*. Acadmic Press, 1980.

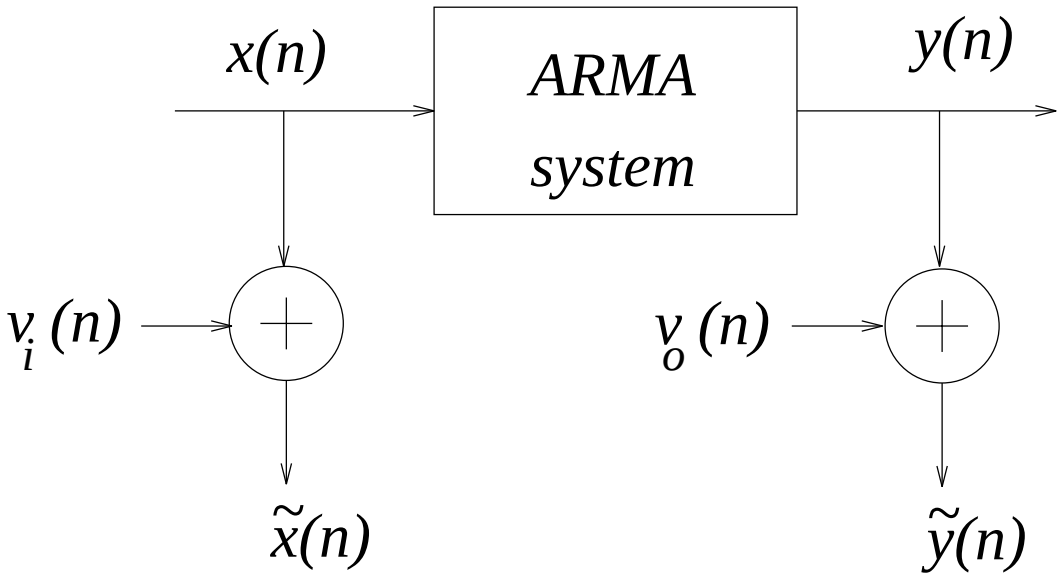

Figure 1: System Identification Problem.

	q16	q17	q18	q19	q20	q21	q22
1	0.0147	0.1891	0.0687	0.3269	0.0074	0.0903	0.3556
2	0.1344	0.3304	0.2125	0.0349	0.0207	0.5408	0.1424
3	0.2232	0.0181	0.2645	0.1537	0.3328	0.2771	0.2956
4	0.1100	0.1725	0.3337	0.0399	0.4746	0.1053	0.1024
5	0.1286	0.2302	0.0708	0.1031	0.3234	0.1476	0.4599
6	0.0860	0.2757	0.1660	0.1969	0.1260	0.3494	0.3260
7	0.0470	0.1349	0.4676	0.0733	0.2745	0.2507	0.1454
8	0.2124	0.3019	0.2626	0.1100	0.4014	0.0610	0.1898
9	0.2419	0.3125	0.0830	0.0087	0.3142	0.2946	0.1893
10	0.1333	0.1592	0.1119	0.2799	0.0582	0.0144	0.2324
11	0.0304	0.1898	0.0734	0.3321	0.0017	0.0927	0.3688
12	0.1036	0.1325	0.2749	0.3650	0.0334	0.4637	0.2107
13	0.3576	0.1674	0.0271	0.4788	0.3161	0.1780	0.1281
14	0.4523	0.1166	0.2749	0.3071	0.1746	0.0703	0.2290
15	0.2169	0.2277	0.0332	0.1372	0.1504	0.0215	0.1106
16	0.1302	0.2921	0.1097	0.2587	0.1309	0.1191	0.0952
17	0.2642	0.0043	0.3457	0.1555	0.1769	0.1909	0.0045
18	0.0110	0.2428	0.1481	0.1985	0.0401	0.0213	0.1358
19	0.3820	0.0821	0.1603	0.0245	0.0033	0.0283	0.0131
20	0.2621	0.3024	0.0343	0.0308	0.0181	0.0140	0.0207
21	0.0330	0.2220	0.1876	0.0493	0.0082	0.0091	0.0145
22	0.2803	0.1574	0.2352	0.0321	0.0027	0.0360	0.0286

Figure 2: zero entries in eigenvectors of $\overline{R}_n$ when $N' - \hat{N} < M' - \hat{M}$.

$\hat{N}$ \ $\hat{M}$	1	2	3	4	5	6	7	8	9	10	11
1	0.0358	0.0319	0.0243	0.0281	0.0399	0.0473	0.0531	0.0541	0.0569	0.0699	0.0592
2	0.0328	0.0342	0.0341	0.0266	0.0303	0.0411	0.0485	0.0521	0.0509	0.0678	0.0560
3	0.0275	0.0272	0.0264	0.0255	0.0238	0.0274	0.0416	0.0439	0.0459	0.0589	0.0635
4	0.0210	0.0212	0.0197	0.0179	0.0231	0.0262	0.0290	0.0377	0.0475	0.0533	0.0529
5	0.0181	0.0147	0.0125	0.0147	0.0142	0.0139	0.0255	0.0297	0.0447	0.0533	0.0376
6	0.0154	0.0140	0.0083	0.0063	0.0066	0.0096	0.0117	0.0273	0.0340	0.0465	0.0361
7	0.0101	0.0096	0.0067	0.0035	0.0006	0.0007	0.0007	0.0008	0.0230	0.0192	0.0442
8	0.0095	0.0070	0.0063	0.0051	0.0017	0.0004	0.0004	0.0004	0.0007	0.0192	0.0271
9	0.0100	0.0085	0.0060	0.0059	0.0049	0.0023	0.0005	0.0005	0.0006	0.0011	0.0160

Figure 3: $\sigma_{\hat{N},\hat{M}}$ Table.

$\hat{N}$ \ $\hat{M}$	1	2	3	4	5	6	7	8	9	10	11
1	1.0000	1.1239	1.3118	0.8647	0.7045	0.8419	0.8916	0.9815	0.9502	0.8145	1.1799
2	1.0904	0.8963	0.7134	1.3566	1.1531	0.8475	0.9291	0.9665	1.1450	0.7730	1.2824
3	1.1950	1.2675	1.3349	1.0768	1.3589	1.3102	0.7651	1.1253	1.0591	0.8988	0.8171
4	1.3096	1.2712	1.4413	1.5722	0.7974	0.9206	1.3012	0.8961	0.7662	0.9867	1.2094
5	1.1596	1.7829	1.8585	1.0269	1.6857	1.9342	0.6166	1.0912	0.7073	0.8371	1.9978
6	1.1739	1.1526	2.5045	3.0564	2.0460	0.9896	1.8004	0.4653	1.0566	0.8385	1.3404
7	1.5284	1.5352	1.7739	3.5462	66.4225	13.3105	13.8747	32.7085	0.0505	2.9032	0.3545
8	1.0626	1.8493	1.1743	0.8288	1.0980	8.0163	1.8752	1.7560	19.0402	0.0367	1.1569
9	0.9454	0.9788	1.5146	0.8807	0.4006	0.3472	3.4924	0.8103	0.9859	10.3573	0.1121

Figure 4: Product of row and column ratios tables.

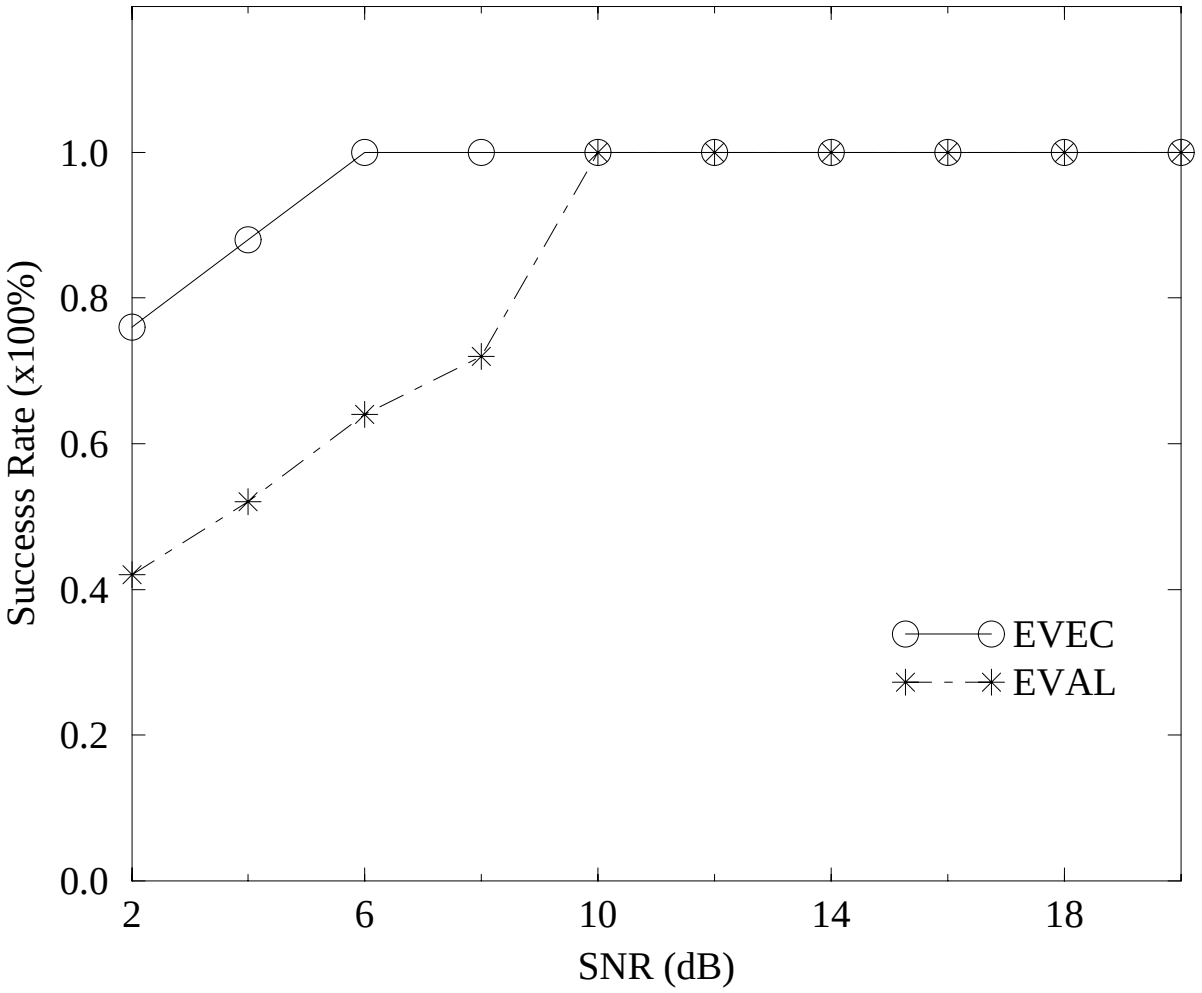

Figure 5: Comparison of eigenvector (EVEC) and eigenvalue (EVAL) methods for *Experiment 1*.

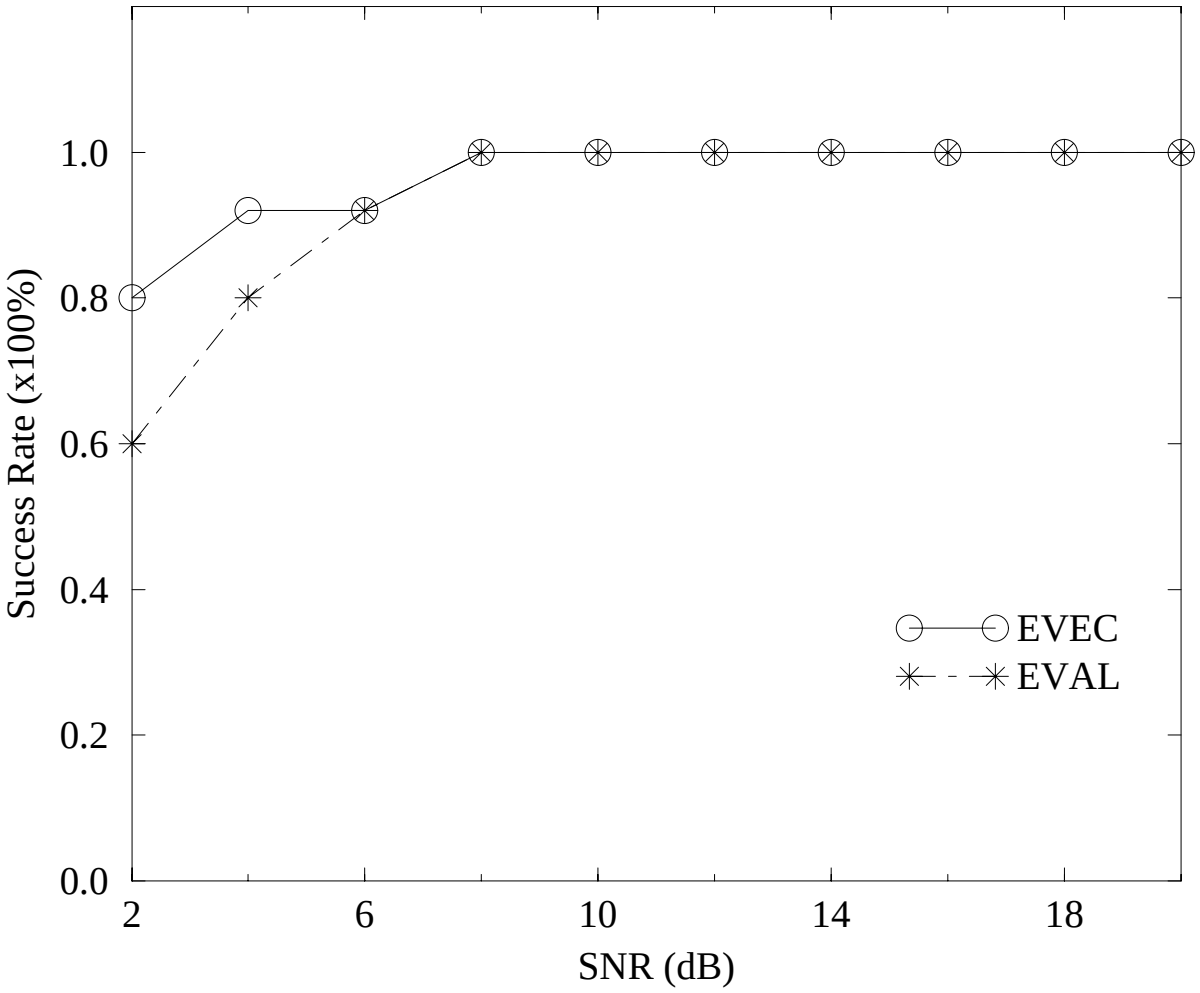

Figure 6: Comparison of eigenvector (EVEC) and eigenvalue (EVAL) methods for *Experiment 2*.