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PROBABILISTIC BOUNDS AND HEURISTIC ALGORITHMS FOR COLORING LARGE RANDOM GRAPHS

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ABSTRACT

Random graphs are of interest in graph algorithm program testing and evaluation, in the assessment of average case graph algorithm behavior, and as a null hypothesis relevant to perceived structure in graph applications in scheduling, resource allocation, and document classification and clustering. To enhance these applications we develop improved lower bounds for the chromatic number of a random graph. We introduce an heuristic estimation formula for the chromatic number of a random graph and a greedy paradigm for a family of coloring algorithms. The proposed algorithms are compared with several heuristic algorithms from the literature and shown to significantly reduce the number of colors required for coloring large random graphs. Our results provide substantial evidence for the conclusion that the chromatic number of a random graph should likely fall in a very narrow range in close proximity to our chromatic number estimate. For the benchmark random graph on one-thousand vertices with edge probability one-half our results indicate that the chromatic number will (with high probability) be in the range 85 ± 12.

Categories and Subject Descriptors: G.2.2 [Discrete Mathematics]: Graph Theory - graph algorithms, G.3 [Probability and Statistics]: Probabilistic Algorithms

General Terms: Algorithms, Design, Experimentation

Additional Key Words and Phrases: graph coloring, random graphs.

I. Introduction and Summary

A coloring of a graph is an assignment of a color to each vertex of a graph where no two vertices joined by an edge receive the same color. The minimum number of colors required to color a graph G is termed the chromatic number $\chi(G)$ of the graph. By a random graph $\underset{\sim}{\mathcal{G}}_{n,p}$ we shall mean a graph on a fixed number n of vertices realized by choosing each pair of vertices to be adjacent (joined by an edge) independently with probability p.

In this paper we develop probabilistic and algorithmic approaches to the computation of bounds on the chromatic number of a random graph. We also investigate the likely distribution of the sizes of sets assigned the same color in minimum (or near minimum) colorings of random graphs both probabilistically and by analysis of the distributions resulting from good heuristic coloring algorithms.

Our work is motivated by the realization that there is considerable interest in the coloring of random graphs arising from a variety of perspectives within computer science. The following instances each indicate the need for knowing the likely value of the chromatic number of a random graph in terms of the size and edge probability parameters.

Algorithm and Program Testing and Evaluation. Random graphs provide a portable family of test graphs that may be readily generated and richly varied in terms of size and edge density. They have frequently been used in the literature for testing general graph coloring algorithms. Random graphs should be a significant component of any benchmark for measuring coloring algorithm performance both in computation time and in terms of accuracy of estimation of the chromatic number by heuristic coloring algorithms.

Average Case Coloring Algorithm Behavior. Random graphs provide a model for assessing average case behavior of coloring algorithms.

Null Hypothesis. A variety of problems in scheduling, resource allocation, and document classification and clustering can be related to the problem of graph coloring. In such applications the occurrences of the edges (related to empirically observed pairwise conflicts/agreements) might have no more structure than that provided by independent random events of fixed probability. If the null hypothesis is associated with the random graph, good information about the possible colorings of random graphs must be available.

In addition to these aforementioned perspectives the relatively large literature on both general graph coloring and general properties of random graphs imbue a sense of fundamental importance to investigation of the colorings of random graphs.

Our study covers graphs of up to a thousand vertices and focuses on the relatively large benchmark random graph $\mathfrak{G}_{1000,1/2}$ having one thousand vertices and approximately a quarter of a million edges. The case $p = 1/2$ is afforded special importance as it dictates equal probability for every possible graph on the given number of vertices. Note that the concept of properties that are said to hold for "almost all graphs" are simply the asymptotic properties of random graphs for $p = 1/2$ with $n \to \infty$.

Any coloring of a graph provides an upper bound on the chromatic number of that graph. To assess how good an approximation to the chromatic number we obtain we must either know or at least have a good lower bound on the chromatic number in question. Utilizing known facts about the probability of a large independent set in a random graph we first derive in Lemma 1 a simple lower bound on the probability that a particular lower bound on the chromatic number must hold. For our benchmark $\underset{\sim}{\mathfrak{G}}_{1000,1/2}$ we thereby obtain that the chromatic number must be at least 67 with probability at least 1 - .032 and at least 63 with probability at least 1 - .000028. Our main result in Section 2 is the derivation of a sharper lower bound computation procedure for random graphs that provides for our benchmark $\underset{\sim}{\mathfrak{G}}_{1000,1/2}$ the improved result that the chromatic number must be at least 73 with probability at least $1 - 1/10^6$ and a least 72 with probability at least $1 - 1/10^{14}$. Tables of lower bounds on the chromatic number of a random graph that must hold with failure at most one in a million are tabulated in Section II.

The joint occurrences of large independent sets in a random graph have been shown [M70] to be relatively weakly correlated events. This fact provides the inspiration for both a "greedy independent set deletion" estimate of the chromatic number and a "greedy independent set deletion" heuristic coloring algorithm which we develop in Section III. The greedy algorithm is shown to color graphs of up to 120 vertices with a number of colors very close to that predicted by the greedy estimate of the chromatic number. The greedy estimate is uniformly quite close to the $(1 - 1/10^6)$-probability lower bound on the chromatic number and grows only slightly faster with increasing n, yielding the estimate 85 (compared to the $(1 - 1/10^6)$-probability lower bound 73) for the chromatic number of our benchmark random graph $\underset{\sim}{\mathfrak{G}}_{1000,1/2}$.

The greedy algorithm of Section III becomes rapidly intractable for graphs over one hundred vertices. In Section IV we apply a variety of sequential coloring algorithms from the literature as well as suboptimal tractable variations of the greedy algorithm to the coloring of our benchmark large random graph $\mathfrak{G}_{1000,1/2}$. Our best proposed algorithm colors a sample of ten randomly generated 1000-vertex edge probability-1/2 graphs within the range 95 to 97 colors, with an average of 95.9 colors. This provides a significant improvement over the best previously reported colorings involving 113 colors [M81] for the random graph $\mathfrak{G}_{1000,1/2}$ employing the DSATUR Algorithm introduced by Brélaz [B79]. From this data we may conclude that the chromatic number of our benchmark random graph $\mathfrak{G}_{1000,1/2}$ is very likely to fall in the range 85 ± 12.

In conclusion the results of this paper provide substantial evidence indicating (i) that the chromatic number of the random graph $\mathfrak{G}_{n,p}$ will with very high probability fall in a quite narrow range, and (ii) that this narrow range should be centered at a value reasonably well predicted by the greedy estimate developed in Section III.

To be reasonably confident of the implementation of the coloring algorithms used for this study a separate checking program was applied to verify the correctness of all colorings generated by all of the coloring algorithms. More detailed tabulations of the results on which average colorings are reported in this paper are given in [J82].

II. Color Partitions and (1-ε)-Probability Lower Bounds

An independent set of the graph G is any set of vertices of G no pair of which are joined by an edge in G, and the independence number $\beta(G)$ denotes the maximum size of any independent set of G. A subgraph of G in which every pair of vertices are joined by an edge of G is a complete subgraph of G, and the clique number $\omega(G)$ denotes the maximum size of the vertex set of any complete subgraph of G. In any coloring of the graph G, all vertices given the same color must form an independent set, so the chromatic number $\chi(G)$ is then equivalent to the minimum number of independent sets which cover the graph G, from which follows the lower bound

$$\chi(G) \geq n/\beta(G) \qquad \text{for any n-vertex graph G.} \tag{1}$$

The vertices of any complete subgraph must all be colored differently, so also

$$\chi(G) \geq \omega(G) \qquad \text{for any graph G.} \tag{2}$$

The random graph $\mathfrak{G}_{n,p}$ shall denote a graph on n vertices realized by choosing each edge to be present independently with probability p, and $q = 1 - p$ is the probability that any pair of vertices is then not adjacent. $\chi_{n,p}$ is the random variable denoting the chromatic number of the random graph $\mathfrak{G}_{n,p}$. $\beta_{n,p}$ and $\omega_{n,p}$ are the random variables denoting respectively the independence number and clique number of the random graph $\mathfrak{G}_{n,p}$.

As noted in [M70], the expected number of independent sets of size j in $\mathfrak{G}_{n,p}$ is given by $\binom{n}{j} q^{\binom{j}{2}}$ where $q = 1 - p$, and the probability of having an independent set of size at least j is at most this value, so

$$\text{Prob}\ \{\beta_{n,p} \geq j\} \leq \binom{n}{j} q^{\binom{j}{2}} \qquad \text{for } 1 \leq j \leq n. \tag{3}$$

From (1) and (3) we can obtain a lower bound on the probability that a particular lower bound on the chromatic number of the random graph must hold.

Lemma 1: For any $n \geq 2$, $j \leq n-1$, with $q = 1-p$, the chromatic number $\underset{\sim}{\chi}_{n,p}$ of the random graph $\underset{\sim}{\mathcal{G}}_{n,p}$ satisfies

$$\text{Prob}\left\{\underset{\sim}{\chi}_{n,p} \geq \frac{n}{j}\right\} \geq 1 - \binom{n}{j+1} q^{\binom{j+1}{2}} . \tag{4}$$

Proof: If $\underset{\sim}{\beta}_{n,p} \leq j$, then by (1) $\underset{\sim}{\chi}_{n,p} \geq n/j$, so

$$\text{Prob}\left\{\underset{\sim}{\chi}_{n,p} \geq \frac{n}{j}\right\} \geq \text{Prob}\,\{\underset{\sim}{\beta}_{n,p} \leq j\}$$

$$= 1 - \text{Prob}\,\{\underset{\sim}{\beta}_{n,p} \geq j+1\}$$

and from (3) the lemma follows. □

Applying Lemma 1 to the random graph $\underset{\sim}{\mathcal{G}}_{1000,1/2}$ we obtain for $j = 15$,

$$\text{Prob}\left\{\underset{\sim}{\chi}_{1000,1/2} \geq \frac{1000}{15}\right\} \geq 1 - \binom{1000}{16}\left(\frac{1}{2}\right)^{120}$$

or

$$\text{Prob}\,\{\underset{\sim}{\chi}_{1000,1/2} \geq 67\} \geq 1 - .032\,,$$

and with $j = 16$

$$\text{Prob}\,\{\underset{\sim}{\chi}_{1000,1/2} \geq 63\} \geq 1 - .000028\,.$$

We now develop a stronger lower bound than that provided by (4) by considering the probability of the existence of a large collection of disjoint j-membered independent sets. Let $i\bar{K}j$ denote a set of i disjoint j-membered independent sets of the graph G. For the random graph $\underset{\sim}{\mathcal{G}}_{n,p}$ let $\| i\bar{K}j \|$ denote the random variable giving the number of $i\bar{K}j$ that occur.

Lemma 2: For n, $j \geq 2$, $i \geq 1$, the expectation of $\| i\bar{K}j \|$ over the random graph $\underset{\sim}{\mathcal{G}}_{n,p}$ is

$$E(\| i\bar{K}j \|) = \frac{n!}{(j!)^i (n-ij)! i!} \, q^{\binom{j}{2} i} . \qquad (5)$$

Proof: The number of ways of partitioning n vertices into i unlabeled sets of size j and one labeled set of size n-ij is $n!/[(j!)^i(n-ij)!i!]$ since the multinomial coefficient $n!/[(j!)^i(n-ij)!]$ must be divided by i! to account for the fact that the i unlabeled sets of size j are indistinguishable among themselves. For each such partition the probability that none of the edges occur between any two vertices of the same j-membered set for every unlabeled j-membered set is $q^{\binom{j}{2} i}$, hence the lemma. □

A k-part partition $P_1, P_2, \ldots, P_k$ of the whole number n denotes a non increasing sequence of positive integers $P_1 \geq P_2 \geq P_3 \geq \ldots \geq P_k \geq 1$ where $\sum_{i=1}^{k} P_i = n$. The partition P_1, P_2, ..., P_k of n is a color partition of the n vertex graph G iff some k-coloring of G has P_i vertices of color i for $1 \leq i \leq k$.

Lemma 3: For n, $j \geq 2$, $i \geq 1$, and the random graph $\underset{\sim}{\mathcal{G}}_{n,p}$ with $q = 1 - p$,

$$\text{Prob}\{P_i \geq j \,|\, P_1, P_2, \ldots, P_k \text{ is a color partition of } \underset{\sim}{\mathcal{G}}_{n,p}\} \leq \frac{n!}{(j!)^i (n-ij)! i!} \, q^{\binom{j}{2} i} . \qquad (6)$$

Proof: For a given $j \geq 2$, $i \geq 1$, suppose some color partition P_1, P_2, ..., P_k of $\underset{\sim}{\mathcal{G}}_{n,p}$ has $P_i \geq j$. Since $P_1 \geq P_2 \geq \ldots \geq P_i \geq j$, $\underset{\sim}{\mathcal{G}}_{n,p}$ must then contain at least one $i\bar{K}j$. Since also Prob{$\underset{\sim}{\mathcal{G}}_{n,p}$ contains at least one $i\bar{K}j$} $\leq E(\| iKj \|)$, the result follows from Lemma 2. □

The right hand side of (6) can be expressed in more manageable form for computation by utilizing Stirling's approximation to n! where for all $n \geq 1$ [R55],

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}} . \qquad (7)$$

Substituting (7) into (6), after some algebra we obtain:

Corollary: For $n, j \geq 2$, $i \geq 1$, $n - ij \geq 1$, and the random graph $\mathfrak{G}_{n,p}$ with $q = 1 - p$,

$$\text{Prob}\{P_i \geq j \mid P_1, P_2, \ldots, P_k \text{ is a color partition of } \mathfrak{G}_{n,p}\}$$

$$< \left[\left[\left(\frac{n}{n-ij}\right)^{\frac{n+1/2}{ij}} \frac{(n-ij)q^{(j-1)/2}}{j}\right]^j \frac{e}{i\sqrt{2\pi j}}\right]^i \frac{1}{\sqrt{2\pi i}\; e^{\frac{1}{12j+1}}} . \qquad (8)$$

Note that the right hand side of (8) can be computed over a considerable range of i,j without danger of overflow. (For further protection from overflow we internally computed the log of the right hand side of (8) in building our subsequent tables.)

We shall first illustrate by an example how Lemma 3 in the form of inequality (8) can be utilized to provide a lower bound on $\underset{\sim}{\chi}_{n,p}$ that must hold with high probability. Specifically we shall show

$\text{Prob}\{\underset{\sim}{\chi}_{1000,1/2} \geq 73\} \geq 1 - 1/10^6$.

Example 1. Consider the following 73-part partition of 1000: $B_1 = 17$, $B_2 = B_3 = 16$, $B_4 = B_5 = \ldots = B_{15} = 15$, $B_{16} = B_{17} = \ldots = B_{45} = 14$, $B_{46} = B_{47} = \ldots = B_{70} = 13$, $B_{71} = B_{72} = 12$, $B_{73} = 2$. Now suppose a particular instance of the random graph $\underset{\sim}{\mathfrak{G}}_{1000,1/2}$ has a chromatic number at most 72. Then there must exist a 72-part color partition $P_1, P_2, \ldots, P_{72}$ of $\underset{\sim}{\mathfrak{G}}_{1000,1/2}$. Since $\sum_{i=1}^{73} B_i = \sum_{i=1}^{72} P_i = 1000$, we must have $P_i > B_i$ for some index i, $1 \leq i \leq 72$. Considering the first index i for which $P_i > B_i$, we conclude that at least one of the following cases must hold: either $P_1 \geq 18$, or $P_2 = 17$, or $P_4 = 16$, or $P_{16} = 15$, or $P_{46} = 14$, or $P_{71} = 13$. Now from inequality (8),

$$\text{Prob}\{P_1 \geq 18\} < 0.012774 \times 10^{-6}$$
$$\text{Prob}\{P_2 = 17\} < 0.000311 \times 10^{-6}$$
$$\text{Prob}\{P_4 = 16\} < 0.009245 \times 10^{-6}$$
$$\text{Prob}\{P_{16} = 15\} < 0.342723 \times 10^{-6}$$
$$\text{Prob}\{P_{46} = 14\} < 0.001687 \times 10^{-6}$$
$$\text{Prob}\{P_{71} = 13\} < 0.000162 \times 10^{-6},$$

from which we determine $\text{Prob}\{\underset{\sim}{\chi}_{1000,1/2} \geq 73\} > 1 - 0.367 \times 10^{-6}$.

Theorem 4: For $n \geq 2$, $p = 1 - q$ and $k \geq 2$, let $B_1, B_2, \ldots, B_k$ be a partition of n with $I = \{i \mid i=1 \text{ or } B_{i-1} > B_i \text{ for } i < k\}$. Let $\delta(i,j) = \frac{n!}{(j!)^i (n-ij)!\,i!} q^{\binom{j}{2} i}$. Then

$$\text{Prob}\{\chi_{n,p} \geq k\} \geq 1 - \sum_{i \in I} \delta(i, B_i+1). \tag{9}$$

Proof: Suppose the random graph $\mathcal{G}_{n,p}$ has a chromatic number less than k. Then there must exist a (k-1)-part color partition $P_1, P_2, \ldots, P_{k-1}$ for $\mathcal{G}_{n,p}$. Since $\sum_{i=1}^{k} B_i = \sum_{i=1}^{k-1} P_i$, we must have $P_j > B_j$ for some $j \in I$. But then $\mathcal{G}_{n,p}$ must have probability at most $\sum_{i \in I} \delta(i, B_i+1)$, so $\text{Prob}\{\chi_{n,p} \geq k\} \geq 1 - \sum_{i \in I} \delta(i, B_i+1)$. □

Given n, p and a probability threshold $(1-\varepsilon)$, we determine the largest value of k, which by Theorem 4 satisfies $\text{Prob}\{\chi_{n,p} \geq k\} \geq 1-\varepsilon$, as follows. Determine the sequence $B_1, B_2, \ldots$ by

$$B_i = \begin{cases} 1 & \text{if } \delta(i,2) < \varepsilon \\ \max\{j \mid \delta(i,j) \geq \varepsilon\} & \text{if } \delta(i,2) \geq \varepsilon \end{cases} \tag{10}$$

until the first index k occurs for which $\sum_{i=1}^{k} B_i \geq n$. Then redefine B_k so that $B_k = n - \sum_{i=1}^{k-1} B_i$, hence $B_1, B_2, \ldots, B_k$ is a k-part partition of n. By Theorem 4 we then obtain $\text{Prob}\{\chi_{n,p} \geq k\} \geq 1 - \sum_{i \in I} \delta(i, B_i+1)$ where $\delta(i, B_i+1) < \varepsilon$ for every $i \in I$. It will generally be the case that $\sum_{i \in I} \delta(i, B_i+1)$ is also less than ε since there are relatively few terms in the sum and since they tend to vary by orders of magnitude. If, however, $\sum_{i \in I} \delta(i, B_i+1) > \varepsilon$, we simply recompute the B_i's using a slightly smaller value ε' in place of ε in (10) until we achieve a partition $B_1, B_2, \ldots, B_k$ for which application of (9) holds with $\sum_{i\varepsilon I} \delta(i,B_i+1)$ at most equal to the original value of ε.

Example 2: Let $n = 1000$, $p = 1/2$, $\varepsilon = 10^{-14}$. The key values in the computation are

$$\delta(1,20) = 0.023547\times 10^{-14}$$

$$\delta(2,18) = 0.005199\times 10^{-14}$$

$$\delta(3,17) = 0.159649\times 10^{-14}$$

$$\delta(7,16) = 0.002526\times 10^{-14}$$

$$\delta(20,15) = 0.245251\times 10^{-14}$$

$$\delta(48,14) = 0.000150\times 10^{-14}$$

$$\delta(72,13) = 0.0000001\times 10^{-14} .$$

We therefore obtain the 72-part partition

$$B_1 = 19,\ B_2 = 17,\ B_3 = B_4 = B_5 = B_6 = 16,\ B_7 = B_8 = \ldots = B_{19} = 15,$$

$$B_{20} = B_{21} = \ldots = B_{47} = 14,\ B_{48} = B_{49} = \ldots = B_{71} = 13,\ B_{72} = 1,$$

and by Theorem 4 confirm that $\text{Prob}\{\chi_{1000,1/2} \geq 72\} \geq 1 - 0.437\times 10^{-14}$.

It is interesting to observe that application of Theorem 4 yielding

$$\text{Prob}\{\underset{\sim}{\chi}_{1000,1/2} \geq 73\} \geq 1 - 0.367\times 10^{-6}$$

$$\text{Prob}\{\underset{\sim}{\chi}_{1000,1/2} \geq 72\} \geq 1 - 0.437\times 10^{-14}$$

as compared to application of Lemma 1 which only confirmed

$$\text{Prob}\{\underset{\sim}{\chi}_{1000,1/2} \geq 67\} \geq 1 - .032$$

$$\text{Prob}\{\underset{\sim}{\chi}_{1000,1/2} \geq 63\} \geq 1 - .000028$$

provides both (i) a higher lower bound with a given probability and (ii) a much sharper threshold below which the chromatic number becomes extremely unlikely.

Employing Theorem 4 we tabulate in Tables 1-3 lower bounds on $\underset{\sim}{\chi}_{n,p}$ that hold with probability $1 - 1/10^6$, i.e. failure at most one in a million, for all n, $50 \le n \le 1000$, with p = .25, .50, and for selected values of n for p = .75.

TABLE NO.1

$(1 - 1/10^6)$-PROBABILITY LOWER BOUNDS ON THE CHROMATIC NUMBER

Edge probability = 0.50

$\text{Prob}\{ \chi \geq \text{Lower bound} \} \geq [1 - 1/10^6]$

Range of no. of vertices.	Lower bound	Range of no. of vertices	Lower Bound
1000 - 997	73	475 - 462	39
996 - 982	72	461 - 448	38
981 - 964	71	447 - 435	37
963 - 947	70	434 - 418	36
946 - 932	69	417 - 404	35
931 - 915	68	403 - 391	34
914 - 898	67	390 - 376	33
897 - 881	66	375 - 361	32
880 - 867	65	360 - 346	31
866 - 850	64	345 - 331	30
849 - 833	63	330 - 318	29
832 - 816	62	317 - 305	28
815 - 801	61	304 - 291	27
800 - 784	60	290 - 278	26
783 - 769	59	277 - 265	25
768 - 753	58	264 - 251	24
752 - 737	57	250 - 239	23
736 - 722	56	238 - 224	22
721 - 707	55	223 - 212	21
706 - 691	54	211 - 199	20
690 - 676	53	198 - 185	19
675 - 660	52	184 - 174	18
659 - 645	51	173 - 163	17
644 - 629	50	162 - 149	16
628 - 613	49	148 - 137	15
612 - 597	48	136 - 125	14
596 - 582	47	124 - 114	13
581 - 566	46	113 - 104	12
565 - 553	45	103 - 92	11
552 - 536	44	91 - 80	10
535 - 520	43	79 - 71	9
519 - 505	42	70 - 60	8
504 - 492	41	59 - 51	7
491 - 476	40	50 - 50	6

TABLE NO.2

$(1-1/10^6)$-PROBABILITY LOWER BOUNDS ON THE CHROMATIC NUMBER

Edge probability = 0.25

$\text{Prob}\{ \underset{\sim}{\chi} \geq \text{Lower bound} \} \geq [1 - 1/10^6]$

Range of no. of vertices.	Lower bound
1000 - 978	35
977 - 945	34
944 - 909	33
908 - 875	32
874 - 838	31
837 - 804	30
803 - 770	29
769 - 737	28
736 - 702	27
701 - 669	26
668 - 636	25
635 - 602	24
601 - 570	23
569 - 537	22
536 - 508	21
507 - 476	20
475 - 444	19
443 - 415	18
414 - 381	17
380 - 354	16
353 - 324	15
323 - 295	14
294 - 265	13
264 - 238	12
237 - 209	11
208 - 183	10
182 - 160	9
159 - 134	8
133 - 107	7
106 - 87	6
86 - 65	5
64 - 50	4

TABLE NO.3

$(1 - 1/10^6)$-PROBABILITY LOWER BOUNDS ON THE CHROMATIC NUMBER

Edge probability = 0.75

$$\text{Prob}\{ \chi \geq \text{Lower bound} \} \geq [1 - 1/10^6]$$

Number of vertices.	Lower bound
1000	132
975	130
950	127
925	124
900	121
875	118
850	115
825	112
800	109
775	106
750	104
725	101
700	98
675	95
650	92
625	89
600	86
575	83
550	80
525	77
500	74
475	71
450	68
425	65
400	61
375	58
350	54
325	51
300	48
275	45
250	41
225	38
200	35
175	31
150	27
125	23
100	19
75	15
50	11

III. Greedy Independent Set Selection Estimate and Coloring Algorithm

In this section we investigate the approach of sequential deletion of maximum independent sets of a graph both to establish an estimate of the chromatic number of the random graph and as a basis for a heuristic coloring algorithm.

The distribution function for $\underset{\sim}{\beta}_{n,p}$, the size of the largest independent set in a random graph, has been shown to have a surprisingly peaked behavior. Matula [M70] derives the following bounds:

$$\begin{aligned} \text{Prob}\{\underset{\sim}{\beta}_{n,p} \geq j\} &\leq \binom{n}{j} q^{\binom{j}{2}}, \\ \text{Prob}\{\underset{\sim}{\beta}_{n,p} \geq j\} &\geq 1 / \sum_{i=\max\{0,2j-n\}}^{j} \frac{\binom{n-j}{j-i}\binom{j}{i}}{\binom{n}{j}} q^{-\binom{i}{2}}. \end{aligned} \tag{11}$$

From these bounds we obtain

$$\text{Prob}\{\beta_{1000,1/2} \geq 15\} \geq 0.8490\,,$$

$$\text{Prob}\{\beta_{1000,1/2} \geq 16\} \leq 0.0319\,,$$

which implies that the random graph $\underset{\sim}{\mathcal{G}}_{1000,1/2}$ will have a largest clique size of 15 with probability greater than 80%. The size of the largest independent set in a random graph is then quite accurately estimated by computing the maximum value of j such that the expected number of independent sets of size j, given by $\binom{n}{j}q^{\binom{j}{2}}$, is at least unity. We may extend this procedure to obtain an estimate of the chromatic number simply by assuming recursive deletion of largest independent sets from the random graph. Formally let $\tilde{\beta}(n)$ denote the estimate of largest independent set size and $\tilde{\chi}(n)$ denote the estimate of the chromatic number of $\mathcal{G}_{n,p}$ where these values are determined for fixed p, with $q = 1 - p$, by

$$\tilde{\beta}(n) = \max\left\{ j \mid \binom{n}{j} q^{\binom{j}{2}} \geq 1 \right\}, \tag{12}$$

$$\tilde{\chi}(n) = 1 + \tilde{\chi}(n - \tilde{\beta}(n)) . \tag{13}$$

Tables 4 and 5 tabulate the values of $\tilde{\beta}(n)$ and $\tilde{\chi}(n)$ for n from 2 to 1000 for $p = 1/2$. Comparison of Tables 1 and 5 show that the chromatic number estimate is not substantially higher than the $(1 - 1/10^6)$-probability lower bounds, representative values being 50 estimated with lower bound 41 for $n = 500$, and 85 estimated with lower bound 73 for $n = 1000$.

To establish credibility for the estimate $\tilde{\chi}(n)$ we shall investigate a corresponding coloring algorithm based on greedy successive deletion of largest independent sets. Note first that the deletion of a largest independent set I of the graph G might tend to increase the density of edges in the remaining graph $G - I$. This effect can be counterbalanced by choosing I among all independent sets of G of size $\beta(G)$ to be a set having largest sum of degrees in G. Thus I is chosen among maximum independent sets to effect maximum edge deletion. We term the following the Exhaustive Edge Deletion (EED) Algorithm.

<u>Algorithm EED</u>: Given a graph $G^* = (V,E)$, this algorithm determines an integer k and a color partition $P_1, P_2, \ldots, P_k$ of the vertices of G^*.

1. [Initialization] Set $k \leftarrow 0$, $G \leftarrow G^*$.
2. [Exhaustive Independent Set Search] Determine a list L containing all independent sets of G of size $\beta(G)$, and set $i \leftarrow 0$.

TABLE NO.4

ESTIMATION OF THE LARGEST INDEPENDENT SET SIZE

Edge probability = 0.50

Range of no.of vertices (n)	$\tilde{\beta}(n)$
1000 - 830	15
829 - 554	14
553 - 369	13
368 - 245	12
244 - 163	11
162 - 108	10
107 - 71	9
70 - 47	8
46 - 31	7
30 - 20	6
19 - 13	5
12 - 8	4
7 - 5	3
4 - 3	2
2 - 1	1

TABLE NO. 5

ESTIMATION OF THE CHROMATIC NUMBER

Edge probability = 0.50

Range of no. of vertices	$\tilde{\chi}(n)$	Range of no. of vertices	$\tilde{\chi}(n)$
1000 - 986	85	410 - 398	43
985 - 971	84	397 - 385	42
970 - 956	83	384 - 372	41
955 - 941	82	371 - 359	40
940 - 926	81	358 - 347	39
925 - 911	80	346 - 335	38
910 - 896	79	334 - 323	37
895 - 881	78	322 - 311	36
880 - 866	77	310 - 299	35
865 - 851	76	298 - 287	34
850 - 836	75	286 - 275	33
835 - 821	74	274 - 263	32
820 - 807	73	262 - 251	31
806 - 793	72	250 - 239	30
792 - 779	71	238 - 228	29
778 - 765	70	227 - 217	28
764 - 751	69	216 - 206	27
750 - 737	68	205 - 195	26
736 - 723	67	195 - 184	25
722 - 709	66	183 - 173	24
708 - 695	65	172 - 162	23
694 - 681	64	161 - 152	22
680 - 667	63	151 - 142	21
666 - 653	62	141 - 132	20
652 - 639	61	131 - 122	19
638 - 625	60	121 - 112	18
624 - 611	59	111 - 102	17
610 - 597	58	101 - 93	16
596 - 583	57	92 - 84	15
582 - 569	56	83 - 75	14
568 - 555	55	74 - 66	13
554 - 541	54	65 - 58	12
540 - 528	53	57 - 50	11
527 - 515	52	49 - 42	10
514 - 502	51	41 - 35	9
501 - 489	50	34 - 28	8
488 - 476	49	27 - 22	7
475 - 463	48	21 - 16	6
462 - 450	47	15 - 11	5
449 - 437	46	10 - 7	4
436 - 424	45	6 - 4	3
423 - 411	44	3 - 2	2

3. [Independent Set Deletion] While L is not empty: Do

   Set $i \leftarrow i+1$ and let I_{k+i} be a set of the list L having maximum sum of degrees in G. Delete I_{k+i} from L and also delete from L any other set of L having a non void intersection with I_{k+i}.

   End.

4. [Is Coloring Complete?] Set $G \leftarrow G - \bigcup_{j=1}^{i} I_{k+j}$, $k \leftarrow k+i$. If G is null, terminate; otherwise go to step 2.

The exhaustive independent set search in step 2 is realized by a standard backtrack search which becomes rapidly intractable for values of $\beta(G)$ above 9 or 10. Since the size of the largest independent set in an n vertex graph has an upper bound of $2 \log_2 n$ for almost all graphs [M72b], [GM75], the EED algorithm has time complexity $n^{O(\log n)}$ for almost all graphs. While not polynomial bounded the EED algorithm is at least subexponential for almost all graphs.

Tables 6 - 9 tabulate results of the application of the EED algorithm for random graphs up to 120 vertices with $p = 1/2$, up to 150 vertices for $p = 3/4$, and up to 40 vertices for $p = 1/4$. The random graphs are created by using a linear congruential pseudo-random number generator with a separate random number chosen to determine the presence or absence of each edge. The averages shown are over a sample of 25 random graphs for the relatively small graphs up to 55 vertices with $p = 1/2$ in Table 6, and over a sample of 10 random graphs in Tables 7 - 9. The computing times indicate the anticipated explosive growth of the algorithm.

The results of these tables indicate that the actual colorings realized by the EED Algorithm are in very close agreement with the estimated chromatic number $\tilde{\chi}(n)$ over the tractable range of application. This provides credibility to the estimate $\tilde{\chi}(n)$ and also suggests Algorithm EED is a good heuristic coloring algorithm when tractable.

TABLE NO. 6

RESULTS OF EED ALGORITHM

Edge probability = 0.50

(Average on sample of 25 random graphs for each n)

n	Average Colors	Min Colors	Max. Colors	Av. Time	$\tilde{\chi}(n)$	Av.Clique Number
5	2.72	2	4	.039	3	2.72
10	4.08	3	5	.111	4	3.88
15	5.21	4	6	.253	5	4.72
20	6.20	5	7	.651	6	5.48
25	6.72	6	8	1.001	7	6.00
30	7.76	7	9	2.981	8	6.44
35	8.72	8	10	3.993	9	6.88
40	9.20	8	10	7.191	9	7.12
45	9.92	9	11	11.155	10	7.28
50	10.68	10	12	20.212	11	7.44
55	11.32	10	13	46.13	11	7.84

Note:
n is the number of vertices
Av.Time is in seconds on a CDC 6600.

TABLE NO. 7

RESULTS OF EED ALGORITHM

Edge probability = 0.50

(Average on sample of 10 random graphs for each n)

n	Average Colors	Min Colors	Max. Colors	Av. Time	$\tilde{\chi}(n)$	$(1-1/10^6)$ L.Bound
60	11.9	11	13	50.31	12	8
70	13.0	12	14	132.11	13	8
80	14.1	13	15	382.76	14	9
90	15.6	15	16	661.32	15	10
100	16.6	16	18	1088.39	16	11
120	18.7	18	20	3492.81	18	13

Note:
n is the number of vertices
Av.Time is in seconds on a CDC 6600.

TABLE NO. 8

RESULTS OF EED ALGORITHM

Edge probability = 0.25

(Average on sample of 10 random graphs for each n)

n	Average Colors	Min Colors	Max. Colors	Av. Time	$\tilde{\chi}(n)$	Av. Largest Clique
10	3.1	3	4	0.12	3	2.9
20	4.6	4	5	1.12	5	4.0
30	5.2	4	6	14.96	5	4.3
40	5.8	5	7	102.21	6	4.5

TABLE NO. 9

RESULTS OF EED ALGORITHM

Edge probability = 0.75

(Average on sample of 10 random graphs for each n)

n	Average Colors	Min Colors	Max. Colors	Av. Time	$\tilde{\chi}(n)$	$(1 - 1/10^6)$ L.Bound
50	16.5	16	17	3.41	16	11
100	26.8	26	29	33.81	26	19
120	30.4	29	31	110.23	29	22
150	36.1	35	37	320.64	34	27

IV. Greedy Vs. Sequential Colorings of Random Graphs

A number of heuristic graph coloring algorithms that have been proposed and analysed in the literature follow the paradigm of sequential coloring as described by Matula et.al. [M72a]. In the sequential coloring paradigm a discipline for ordering of the vertices is first prescribed and then each vertex in this order is assigned the minimum possible color which has not yet been assigned to any previously colored adjacent vertex. Four sequential coloring algorithms which have formed the basis of several comparative studies in the literature are characterized by the following vertex orderings:

Random Vertex Ordering	(RANDOM Algorithm)
Largest First Vertex Ordering	(LF Algorithm)
Smallest Last Vertex Ordering	(SL Algorithm)
Degree Staturation Vertex Ordering	(DSATUR Algorithm)

Random vertex ordering has been utilized in several studies [M72a], [D77], [L79], [M82] as a basis for comparison with other vertex orderings. The probabilistically simple structure of random ordering makes the resulting RANDOM sequential coloring algorithm susceptible to asymptotic analysis [GM75].

In the LF Algorithm the vertices are ordered so that their degrees form a non increasing sequence. This algorithm was introduced by Welsh and Powell [WP67], also studied in [C71], [B79], [L79], and was termed the largest first algorithm in [M72a].

In the SL Algorithm the vertices are arranged in an order $v_1, v_2, \ldots, v_n$ so that vertex v_i has smallest degree in the subgraph determined by the vertices $\{v_1, v_2, \ldots, v_i\}$ for each i. Note that such an order is readily computed by determining (and deleting) the vertices in the order $v_n, v_{n-1}, v_{n-2}, \ldots, v_1$. The SL Algorithm was introduced by Matula et.al. [M72a] and has also been studied in [B79], [L79] . The SL Algorithm represents an improvement over the LF Algorithm in its guaranteed bound, however the performance of the LF and SL Algorithms on random graphs is very similar.

The DSATUR algorithm given by Brélaz [B79] is based on dynamic vertex ordering in which each vertex is selected at the time it is to be colored. The saturation degree of a vertex is defined as the number of different colors assigned to vertices adjacent to the given vertex. Initially a vertex with maximum degree is given the first color. Among all the uncolored vertices a vertex with maximum saturation degree is then selected (if there is a tie then choose among these a vertex of maximum degree in the uncolored graph). The selected vertex is then given the minimum possible color which has not yet been given to any of its adjacent vertices. This procedure is repeated until all the vertices are colored.

The RANDOM, LF, and SL sequential coloring algorithms can be implemented in time $O(|V| + |E|)$ where $|V|$ is the number of vertices and $|E|$ is the number of edges in the given graph, although we do not pursue the complexity issue here.

The RANDOM Algorithm asymptotically colors a random graph G with at most $(2 + \varepsilon)\ \chi(G)$-colors for any $\varepsilon > 0$ for all graphs except an arbitrarily small fraction as $|V| \to \infty$, as shown by Grimmett and McDiarmid [GM75]. In practice the other sequential algorithms perform somewhat better but their asymptotic behavior is not known. Note that the worst case behavior of all these algorithms is very bad as has been shown by Johnson [J74] and Mitchem [M76].

The performance of these four sequential coloring algorithms for a sample of 10 random graphs with edge probability $p = 1/2$ and $n = 60, 70, \ldots, 120$ in comparison with the EED Algorithm, the estimated chromatic number, and the $(1 - 1/10^6)$-lower bound is shown in Table 10. These results indicate a slight progressive deterioration in the effectiveness of the sequential coloring algorithms with increasing n.

The sequential coloring algorithms are remarkably efficient and can be applied to large random graphs. Table 11 tabulates the results of application of these algorithms to a sample of 10 1000-vertex edge probability 1/2 random graphs. It should be noted that the deviation between minimum and maximum number of colors required over the sample of 10 random graphs by each algorithm is quite small, allowing a ranking of the effectiveness of the algorithms.

The stability of the results evident in Table 11 adds credibility to the "portability" of random graphs as test graphs for coloring algorithms. Durre [D77] reports application of the RANDOM, LF and SL Algorithms for a random graph with $n = 1000$ and $p = 1/2$, achieving 129, 123 and 122 colors respectively. Manvel [M82] uses a slightly different concept of random graph obtaining 125 colors for RANDOM, 120 for LF and 113 for DSATUR applied to such a random graph for $n = 1000$ and $p = 1/2$. The stability of the results are encouraging, but the gap between the best colorings both in Table 11 and the literature for the random graph $\mathcal{G}_{1000,1/2}$ and the estimate $\tilde{\chi}_{1000,1/2} = 85$ leads us to consider an improvement in the sequential coloring paradigm.

All the sequential coloring algorithms can be improved simply by applying recursiveness to the algorithms as follows. Once a sequential coloring of a graph is obtained the vertices which form the bigger color-classes are chosen and those vertices are deleted from the graph. The algorithm is then reapplied to the remaining subgraph, and the process is continued recursively. If we

TABLE NO.10

SEQ. AND EED COLORINGS Vs. ESTIMATED CHROMATIC NUMBER AND $(1-1/10^6)$-LB

(Average on sample of 10 random graphs for each n)

Edge probability = 0.50

n	RANDOM	LF	SL	DSATUR	EED	$\tilde{\chi}(n)$	$(1-1/10^6)$-LB
60	14.7	13.4	13.8	12.5	11.9	12	8
70	16.1	15.1	15.5	14.0	13.0	13	8
80	18.3	16.7	17.1	15.3	14.1	14	9
90	19.9	18.3	18.9	16.8	15.6	15	10
100	21.2	19.9	20.0	18.2	16.6	16	11
120	24.7	22.9	23.5	20.8	18.7	18	13

TABLE NO.11

RESULTS OF SEQ. COLORING ALGORITHMS FOR LARGE GRAPHS

(Average on sample of 10 random graphs)

Number of vertices = 1000

Edge probability = 0.50

Algorithms	Average Colors	Min. Colors	Max. Colors	Average Time
RANDOM	127.3	126	129	100.14
LF	122.7	121	125	101.31
SL	124.3	122	127	126.73
DSATUR	115.8	115	117	111.36
$\tilde{\chi}(1000)$	85	-	-	-
$[1-1/10^6]$-LB	73	-	-	-

Note:

Average time is in seconds on a CDC 6600 computer.

$[1-1/10^6]$-LB denotes a lower bound on the chromatic number that holds with probability at least $1-1/10^6$.

guarantee that in each step a fixed fraction of vertices are removed from the graph then the recursive improvement can be implemented so as to preserve $O(|V| + |E|)$-time for the RANDOM, LF, and SL sequential algorithms, but again we do not pursue this issue here.

Recursive improvement versions of the four sequential algorithms are tested for 1000-vertex graphs with edge probability 1/2 in Table 12. The recursive sequential algorithms give a notable improvement over the straight sequential algorithms and also provide stability over the sample.

The Exhaustive Edge Deletion (EED) algorithm is intractable for these larger graphs. We have seen in Section III that over 100 vertices with edge probability 0.50 the process becomes rapidly inefficient.

<u>2-Phase Greedy/Exhaustive Algorithms</u>

A tractable variation of the EED Algorithm for large random graphs is provided by the following Greedy/Exhaustive procedure. By this procedure the process of selecting a large independent set of a graph for deletion is accomplished in two phases. The first phase involves the selection of an independent set I_1 of the current graph G large enough simply to assure that the remaining set R of vertices of G non adjacent to any member of I_1 is less than some threshold T. The second phase involves application of the EED Algorithm to the subgraph on the vertices R using as degree weights the degrees of the vertices of R in G, determining an independent set $I_2 \subset V^*$. $I_1 \cup I_2$ is then the desired independent set selection from G. The threshold T is set only so large as to make the whole computation tractable.

We shall now describe and tabulate results on applications of three specific Greedy/Exhaustive Algorithms. These three algorithms are progressively more complex and yield progressively better colorings. The first

TABLE NO.12

RESULTS OF DIFFERENT COLORING ALGORITHMS

(Average on sample of 10 random graphs)

Number of vertices = 1000
Edge probability = 0.50

Algorithms	Average Colors	Min. Colors	Max. Colors	Average Time
RANDOM	127.3	126	129	100.14
LF	122.7	121	125	101.31
SL	124.3	122	127	126.73
DSATUR	115.8	115	117	111.36
RANDOM-REC	116.9	115	119	310.71
LF-REC	115.7	114	117	231.79
SL-REC	115.0	113	117	540.11
DSATUR-REC	111.4	110	113	498.38
GE1	105.2	104	107	432.81
GE2	100.1	99	101	1128.24
GE3	95.9	95	97	3212.17
$\tilde{\chi}(1000)$	85	-	-	-
$[1-1/10^6]$-LB	73	-	-	-

Note:
REC represents the recursive version.
Average time is in seconds on a CDC 6600.
$[1-1/10^6]$-LB denotes a lower bound on the chromatic number which holds with probability at least $1-1/10^6$.

procedure involves straightforward random selection in phase 1 and is given as a cononical reference for the Greedy/Exhaustive procedure.

Algorithm GE1. Given a graph $G^* = (V^*, E^*)$ and a threshold T, this algorithm determines an integer k and a color partition $P_1, P_2, \ldots, P_k$ of the vertices of G^*.

1. [Initialize Coloring] $k \leftarrow 1$, $V \leftarrow V^*$, $G \leftarrow G^*$.
2. [Initialize Next Ind. Set Selection] $P_k \leftarrow \emptyset$, $R \leftarrow V$.
3. [Phase 1 Ind. Set Selection] While $|R| > T$, choose v randomly from R, set $P_k \leftarrow P_k \cup \{v\}$, $R \leftarrow R - \{v\} - \{u | u \text{ adjacent to } v \text{ in } G^*\}$.
4. [Phase 2 Ind. Set Selection] Apply the EED Algorithm to the subgraph of G on the vertex set R using as degree weights the degrees in G of the vertices in R, denoting by I the resulting independent set determined from R. Set $P_k \leftarrow P_k \cup I$.
5. [Finished?] Set $V \leftarrow V - P_k$, $G \leftarrow G - P_k$. Then if $V = \emptyset$ terminate, otherwise set $k \leftarrow k+1$ and go to step 2.

Our second Greedy/Exhaustive Algorithm uses the concept of "sampling" and is termed Algorithm GE2. A desired independent set size is prespecified. An independent set is selected as in steps 3 and 4 of Algorithm GE1, but the set is made part of the color partition only if it is at least as large as the prespecified size. If the selected independent set is smaller than the prespecified size it is rejected and the search is repeated for another independent set. If rejection occurs a specified number of times in succession, the desired independent set size is decremented by unity and the process repeated. A step-by-step description of Algorithm GE2 can be readily developed from the description of Algorithm GE1 simply by incorporating the above described sampling loop.

Algorithm GE3 employs a number of additional heuristic criteria to enhance the performance of the Greedy/Exhaustive procedure. For brevity the main ideas will only be outlined informally by comparison with the steps of Algorithm GE3. The vertex v rather than being chosen randomly as in step 3 of GE1 is chosen in Algorithm GE3 so that the set $R - \{v\} - \{u|u \text{ adjacent to } v\}$ will be larger than average without having an appreciably higher density of edges in the subgraph on those vertices. This goal is implemented by first choosing several vertices of R of relatively small degree and then choosing for selection the one of these having largest average degree on its adjacent vertices. To improve the performance over Algorithm GE1 for selecting the tail $P_i, P_{i+1}, \ldots, P_k$ of the color partition, Algorithm GE3 incorporates the EED Algorithm to complete this selection after some appropriate threshold.

The three Greedy-Exhaustive Algorithms are applied to the same 10 random graphs having $n = 1000$, $p = 1/2$ as were the preceding sequential coloring algorithms and the results for all algorithms are tabulated in Table 12 for easy comparison. A threshold of $T = 44$ was used for phase 2 of the GE algorithms. For Algorithm GE2 the prespecified desired independent set size was initially set at 13, with the level of number of rejections before lowering the independent set size kept equal to the desired independent set size.

Note that the "canonical" Greedy/Exhaustive Algorithm GE1 yielded considerably improved colorings over any of the sequential or recursive sequential algorithms in reasonably comparable time. Separate application of Algorithm GE1 with the threshold set higher at $T = 64$ yielded colorings averaging 101.9 colors with average execution time increasing to 2418.4 seconds. This indicates the sampling procedure incorporated in Algorithm GE2 is more cost effective than an increase in the threshold level of GE1.

The various enhancements utilized in Algorithm GE3 provide a further substantial improvement in minimizing the number of colors needed for coloring this sample of random graphs. The techniques employed in Algorithm GE3 were further analysed and seen to preserve the goal of maintaining the average edge density in the remaining graph at about 50% even as somewhat larger independent sets were deleted. For the tail of the original $n = 1000$, $p = 1/2$ graphs when the residual subgraph to be colored first had less than eighty vertices the average edge density was noted to be 0.5192 over the sample of 10 graphs by Algorithm GE3.

The average color partitions generated by the algorithms for the same ten graph sample of Table 12 are shown in Table 13. The three Greedy/ Exhaustive Algorithms are seen to successively give color partitions more closely approaching that of the chromatic number estimate. None of the sequential algorithms was able to find a color-class of size 14, but the GE3 Algorithm was able to find several independent sets of size 14 and once found an independent set of size 15. As previously noted this is likely to be a largest independent set with probability greater than 80% for that graph. The data gives considerable credibility to the conjecture that application of Algorithm EED to the same graphs (if tractable) would yield colorings requiring only about 85-88 colors in accord with the estimate $\tilde{\chi}(1000) = 85$.

A final conclusion relevant to testing general heuristic graph coloring algorithms should be noted. The random graph $\mathcal{G}_{1000,1/2}$ clearly represents a very portable and non trivial application for any such algorithm, where an actual chromatic number of 85 ± 12 may be assumed to hold with great confidence.

TABLE 13

AVERAGE COLOR-PARTITIONS OBTAINED BY DIFFERENT ALGORITHMS

Number of Vertices = 1000; Edge probability = 0.50
(Average is on sample of 10 random graphs for each Algo.)

Size \ Algo.	RD	LF	SL	DS	RD-R	LF-R	SL-R	DS-R
17	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
13	0.0	0.0	0.1	0.0	0.0	0.0	0.2	0.0
12	0.5	0.2	0.4	0.4	1.0	0.2	0.4	2.2
11	4.1	2.3	4.0	5.4	13.4	7.4	9.9	14.7
10	17.3	19.7	19.5	25.2	30.7	27.4	31.4	33.2
9	33.1	38.4	37.9	39.5	26.6	36.3	31.2	29.3
8	30.2	33.0	25.7	26.6	19.3	23.7	19.6	15.1
7	16.7	12.6	15.5	11.1	9.3	10.7	11.1	7.2
6	10.8	6.8	8.8	4.2	6.6	5.2	5.0	3.5
5	5.9	4.0	5.1	1.6	3.0	2.6	2.9	2.7
4	3.6	2.0	2.8	0.5	2.6	0.9	1.5	1.5
3	2.0	2.0	2.1	0.5	1.9	0.8	0.8	0.7
2	1.7	1.0	1.5	0.3	1.5	0.3	0.5	1.1
1	1.4	0.7	0.9	0.5	1.0	0.2	0.5	0.2

Size \ Algo.	GE1	GE2	GE3	$\tilde{\chi}(n)$	$(1-1/10^6)$-LB
17	0.0	0.0	0.0	0.0	1.0
16	0.0	0.0	0.0	0.0	2.0
15	0.0	0.0	0.1	12.0	12.0
14	0.0	0.0	0.8	20.0	30.0
13	0.6	1.4	19.8	14.0	25.0
12	6.6	20.4	22.7	10.0	2.0
11	29.0	33.5	16.4	7.0	0.0
10	29.8	18.4	10.6	6.0	0.0
9	16.9	8.9	7.7	4.0	0.0
8	9.0	5.7	4.6	3.0	0.0
7	5.1	2.7	3.0	2.0	0.0
6	2.8	3.2	3.1	2.0	0.0
5	1.5	2.0	2.8	1.0	0.0
4	1.7	1.5	1.4	1.0	0.0
3	1.0	0.7	1.0	1.0	0.0
2	0.9	0.9	0.9	1.0	1.0
1	0.3	0.8	1.0	1.0	0.0

Note:
RD - RANDOM ; LF - Largest First ; SL - Smallest Last
DS - Degree Saturation ;
-R represents the recursive version.
Algo represents the term for algorithms.
Size represents the no. of vertices in the parts of the color-partition.

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