CSE2353 - BOOLEAN ALGEBRA OUTLINE

- What is a Boolean Algebra?
- Relationship to Sets and Propositions
- Boolean Functions
- Minterms/Maxterms
- Boolean Algebra in Computer Science

INTRODUCTION

$(\{0,1\},+,*,^{-},0,1)$

$+$	0	1
0	0	1
1	1	1

$*$	0	1
0	0	0
1	0	1

b	$\overline{\mathrm{b}}$
0	1
1	0

$(P(\{x\}),\cup,\cap,^{-},\emptyset,\{x\})$

$\cup$	$\emptyset$	$\{x\}$
$\emptyset$	$\emptyset$	$\{x\}$
$\{x\}$	$\{x\}$	$\{x\}$

$\cap$	$\emptyset$	$\{x\}$
$\emptyset$	$\emptyset$	$\emptyset$
$\{x\}$	$\emptyset$	$\{x\}$

b	$\overline{\mathrm{b}}$
$\emptyset$	$\{x\}$
$\{x\}$	$\emptyset$

$(\{f,t\},\vee,\wedge,^{-},f,t)$

$\vee$	f	t
f	f	t
t	t	t

$\wedge$	f	t
f	f	f
t	f	t

b	$\overline{\mathrm{b}}$
f	t
t	f

WHAT IS A BOOLEAN ALGEBRA?

- $(B, +, *, ^{-}, \mathbf{0}, \mathbf{1})$
- Binary Operator + (Sum)
- Binary Operator $*$ (Product)
- Unary Operator $^{-}$ (Complement)
- $\mathbf{0}$ (Identity): $\forall b \in B, b + \mathbf{0} = \mathbf{0} + b = b$
- $\mathbf{1}$ (Identity): $\forall b \in B, b * \mathbf{1} = \mathbf{1} * b = b$
- + and $*$ are Associative
- + and $*$ are Commutative
- + Distributes over $*$
- $*$ Distributes over +
- $\forall b \in B, b + \overline{b} = \mathbf{1}$
- $\forall b \in B, b * \overline{b} = \mathbf{0}$

BOOLEAN ALGEBRA PROPERTIES

- Dual - Change Binary operators and Identities
- The dual of a Boolean algebra theorem is a Boolean algebra theorem
- The identity elements are unique
- The complement of an element is unique
- $\forall b \in B, b + b = b$ and $b * b = b$
- $\forall b \in B, \mathbf{1} + b = \mathbf{1}$ and
  $\mathbf{0} * b = \mathbf{0}$
- Absorption: $\forall b_1, b_2 \in B, b_1 + (b_1 * b_2) = b_1$
  and $b_1 * (b_1 + b_2) = b_1$
- Involution: $\forall b \in B, \overline{\overline{b}} = b$
- DeMorgan's Laws: $\forall b_1, b_2 \in B, (\overline{b_1 + b_2}) = \overline{b_1} * \overline{b_2}$
  and $(\overline{b_1 * b_2}) = \overline{b_1} + \overline{b_2}$

BOOLEAN EXPRESSIONS

- Given (B,+,$*$,$^-$,**0**,**1**), a Boolean Variable is a variable over the set B
- A Literal is a Boolean variable, x, or its complement $\bar{x}$
- Boolean Expressions
  - Identity Elements **0, 1**
  - Boolean Variables $x_1, x_2, ..., x_n$
  - $(X + Y), (X * Y), \overline{X}$ where X and Y are Boolean Expressions
- Two Boolean expressions are equivalent (equal) if one can be obtained from the other by a finite sequence of applications of the Boolean algebra axioms

BOOLEAN FUNCTIONS

- A Boolean Function f:$B^n \rightarrow B$ such that f$(x_1, x_2, ..., x_n)$ is a Boolean expression
- Examples: $f_1(x_1, x_2) = x_1 + x_2$; $f_2(x_1, x_2, x_3) = \overline{x_1} * (x_2 + \overline{x_3})$
- Notation: Use $x_1 x_2$ to mean $x_1 * x_2$
- Precedence Order: Complement, Product, Sum, Left to Right

BOOLEAN FUNCTION ON ONE VARIABLE

- f(x)=**1**, g(x)=x
- Examine f(x)+f(x), f(x)+g(x), f(x)g(x), g(x)+g(x), g(x)g(x), $\overline{f(x)}$,$\overline{g(x)}$,$\overline{(}f(x))$
- Any function f(x) can be written as f(x)=f(**0**)x+f(**1**)$\overline{x}$.
- **0**x+**0**$\overline{x}$ = **0**
- **0**x+**1**$\overline{x}$ = $\overline{x}$
- **1**x+**0**$\overline{x}$ = x
- **1**x+**1**$\overline{x}$ = x+$\overline{x}$ = **1**

MINTERMS & MAXTERMS

- One variable: x,$\overline{x}$
- Two variables: x_1x_2, $x_1\overline{x_2}$, $\overline{x_1}x_2$, $\overline{x_1x_2}$
- Minterm on n variables $x_1, x_2, ..., x_n$ is a Boolean expression which has the form of the product of each Boolean variable or its complement.
- Two minterms on one variable, four on two, eight on 3, and 2^n on n.
- Notation: $x^0 = \overline{x}$ and $x^1 = x$.
- $m_{e_1e_2...e_n} = x_1^{e_1}x_2^{e_2}...x_n^{e_n}$
- Examples: $m_{11} = x_1x_2$, $m_{10} = x_1\overline{x_2}$, $m_{01} = \overline{x_1}x_2$, $m_{00} = \overline{x_1x_2}$
- Maxterm on n variables $x_1, x_2, ..., x_n$ is a Boolean expression which has the form of the sum of each Boolean variable or its complement.
- Examples: $M_{11} = x_1 + x_2$, $M_{10} = x_1 + \overline{x_2}$, $m_{01} = \overline{x_1} + x_2$, $m_{00} = \overline{x_1 + x_2}$

CANONICAL FORMS OF BOOLEAN EXPRESSIONS

- Any Boolean expression can be written as the sum of minterms (or product of maxterms)
- Disjunctive Normal Form - Boolean expression written as sum of minterms
- Conjuctive Normal Form - Boolean expression written as product of maxterms
- We will normally use DNF
- To convert a Boolean expression, $f(x_1, x_2, ..., x_n)$ into DNF, we need to determine the values for the constant prefixes (That is the $f(e_1, e_2, ..., e_n)$).
- Example: Convert $x_1 + x_2$ into DNF

CONVERTING TO CANONICAL FORM

- Algorithm:

  1. Create table showing values of $f(e_1, ..., e_n)$
  2. Rewrite f using the output values shown as the prefixes for the corresponding terms

- Example:

e_1	e_2	$f(e_1, e_2)$
0	**0**	**0**
0	**1**	**0**
1	**0**	**1**
1	**1**	**1**

GENERAL BOOLEAN ALGEBRAS

- The Power Set for any finite set can be used to define a Boolean Algebra
- Example: Look at A={x,y}
- The cardinality of a finite Boolean algebra is a power of 2
- In Computer Science we are primarily interested in the Boolean algebra of cardinality 2

BOOLEAN ALGEBRA IN COMPUTER SCIENCE

- Switching Circuits
- Logic Networks
- Karnaugh Maps

SWITCHING CIRCUITS

- Two-state Device: On or Off (1 or 0; T or F)
- Switch - Open (No current) or Closed (Current)
- Switches Can be combined:
  - Parallel - Boolean algebra +
  - Series - Boolean algebra $*$
  - If S denotes a switch then $\overline{\mathrm{S}}$ denotes a switch which always has the opposite state
- Switching functions can be created to represent any Boolean function
- Switches can be mechanical or electronic

LOGIC NETWORKS

- Switching circuits either conduct (switch closed) or do not conduct (switch open) electricity. For example voltage-operated circuits could define **1** as a signal with ≥ 3 volts and **0** with 0 volts and an acceptable tolerance
- Gate - hardware that implements logic operations of AND, OR, NOT (inverter)
- Linking gates together creates a logic network
- Adders