Boolean Algebra

- Mathematical Model for Describing Digital Circuits
- Circuits are Electronic – Voltage/Current Levels
- Bi-stable Circuits (as opposed to Analog)
- Assign 0 and 1 to Each Bi-stable Value
- Allows use of Binary-valued Boolean Algebra to Describe Circuits

What is an Algebra?

- A Collection of "Things"
  1) ***Set*** of Elements (numbers)
  2) Set of Operators (*unary* and/or *binary*)
  3) Collection Obeys Certain Rules

Example: Algebra over Real-valued Numbers:
- **Elements** – All Real Numbers
- **Operators** – Addition, Subtraction, Multiplication, Division, Inversion, etc.
- "**Certain Rules**" allow some operators to be defined based on others

What are the "Rules"?

1) Closure
   - After Operation Result is in Set
2) Associativity $(x*y)*z = x*(y*z)$
3) Commutativity $x*y = y*x$
4) Identity Element(s) $e*x = x*e = x$
5) Inverse(s) $x*y = e$
6) Distributivity $x*(y \bullet z) = (x*y) \bullet (x*z)$

Boolean Algebra

- Set of elements B
  (B contains at least two distinct elements)
- Two Binary Operators (Closure over B)
  1) +
  2) •
- B Contains Identities wrt + and • (0 and 1)
- Commutativity wrt + and •
- Distributivity wrt + and •
- Inverse for all Elements of B (Complements)

Operation Truth Tables

x	y	$x \bullet y$
0	0	0
0	1	0
1	0	0
1	1	1

AND

x	y	$x+y$
0	0	0
0	1	1
1	0	1
1	1	1

OR

x	$\overline{x}$
0	1
1	0

NOT

Distributivity Example

$$f = x \bullet (y + z)$$

$$f = (x \bullet y) + (x \bullet z)$$

x	y	z	$y + z$	$x \cdot (y + z)$	$x \cdot y$	$x \cdot z$	$(x \cdot y) + (x \cdot z)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

Another Distributivity Example

$$f = x + (y \bullet z)$$
$$f = (x + y) \bullet (x + z)$$

x	y	z	$x+y$	$x+z$
0	0	0	0	0
0	0	1	0	1
0	1	0	1	0
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

x	y	z	$x+yz$	$(x+y)(x+z)$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

NOTE: This Doesn't Work for Algebra over Real Numbers!

Huntington Postulates and Theorems

Table 2-1
Postulates and Theorems of Boolean Algebra

Postulate 2	(a)	$x + 0 = x$	(b)	$x \cdot 1 = x$
Postulate 5	(a)	$x + x' = 1$	(b)	$x \cdot x' = 0$
Theorem 1	(a)	$x + x = x$	(b)	$x \cdot x = x$
Theorem 2	(a)	$x + 1 = 1$	(b)	$x \cdot 0 = 0$
Theorem 3, involution		$(x')' = x$		
Postulate 3, commutative	(a)	$x + y = y + x$	(b)	$xy = yx$
Theorem 4, associative	(a)	$x + (y + z) = (x + y) + z$	(b)	$x(yz) = (xy)z$
Postulate 4, distributive	(a)	$x(y + z) = xy + xz$	(b)	$x + yz = (x + y)(x + z)$
Theorem 5, DeMorgan	(a)	$(x + y)' = x'y'$	(b)	$(xy)' = x' + y'$
Theorem 6, absorption	(a)	$x + xy = x$	(b)	$x(x + y) = x$

Operator Precedence

1) Expressions in Parenthesis
2) NOT (complement) operations
   - Overbar Notation IMPLIES Parenthesis
3) AND Operation
   - Absence of • Between Two Variables Implicit
4) OR Operation
   - Denoted by +

Boolean Functions

1) Can Describe in Many Ways
   - Truth Tables
   - Symbolic (algebraic) Notation
   - RTL Descriptions
   - Graphs
2) Range – 1 or more Values in B
3) Domain – 1 or more values in B

Boolean Function Example

x	y	z	F_1
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

$$F_1 = x + \overline{y}z$$

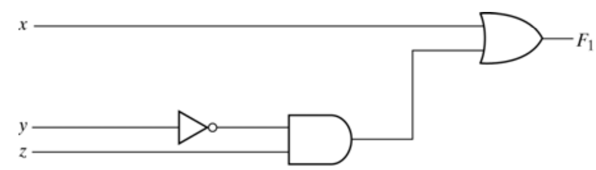
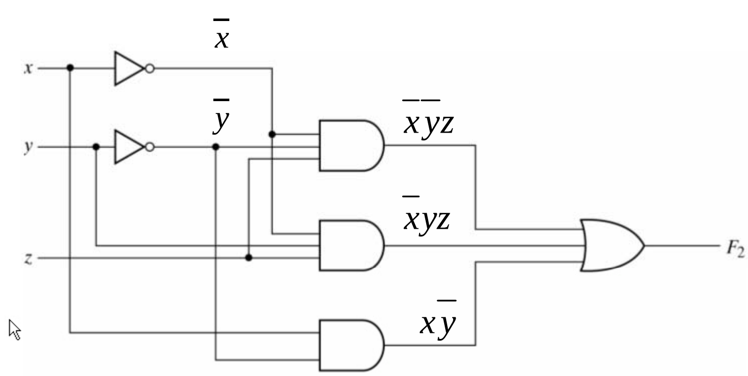


Fig. 2-1 Gate implementation of $F_1 = x + y'z$

Function Describes Circuit



(a) $F_2 = x'y'z + x'yz + xy'$

Function Describes Circuit

$$F_2 = \overline{x}\,\overline{y}z + \overline{x}yz + x\overline{y}$$

x	y	z	$x'y'z$	$x'yz$	xy'	F_2
0	0	0	0	0	0	0
0	0	1	1	0	0	1
0	1	0	0	0	0	0
0	1	1	0	1	0	1
1	0	0	0	0	1	1
1	0	1	0	0	1	1
1	1	0	0	0	0	0
1	1	1	0	0	0	0

Algebra to Simplify

$$f = \overline{x}\,\overline{y}z + \overline{x}yz + x\overline{y}$$

$$f = \overline{x}(\overline{y}z + yz) + x\overline{y}$$

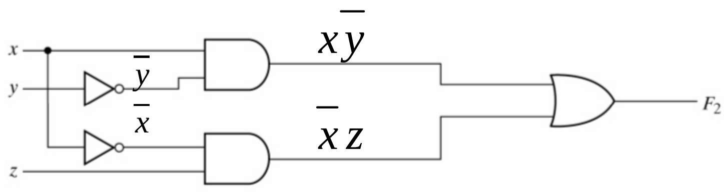
$$f = \overline{x}[z(\overline{y} + y)] + x\overline{y}$$

$$f = \overline{x}[z(1)] + x\overline{y}$$

$$f = \overline{x}z + x\overline{y}$$

Simplified Circuit

$$F_2 = \overline{x}z + x\overline{y}$$



(b) $F_2 = xy' + x'z$