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An Approximate Fourier Transform Useful in Quantum Factoring

D. Coppersmith

IBM Research Division
T.J. Watson Research Center
Yorktown Heights, New York

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Don Coppersmith

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Abstract. We define an approximate version of the Fourier transform on 2^L elements, which is computationally attractive in a certain setting, and which may find application to the problem of factoring integers with a quantum computer as is currently under investigation by Peter Shor. [SHO]

Fourier Transform

Notation: Let L be a positive integer. Let a, c be L-bit integers. The binary representations of a, c are

$$a = \Sigma_{i=0}^{L-1} a_i 2^i, c = \Sigma_{i=0}^{L-1} c_i 2^i.$$

Define the L-bit integer b as the reversal of c,

$$b = \Sigma b_i 2^i = \Sigma c_{L-1-i} 2^i,$$

so that $b_i = c_{L-1-i}$. Let X, Y be arrays of size 2^L indexed by a or c. Let $\omega = \omega_{(2^L)} = \exp\left(2\pi i / 2^L\right)$ be the standard 2^L root of unity.

The ordinary Fourier transform is defined as

$$Y_c = \frac{1}{\sqrt{2^L}} \Sigma_a X_a \omega^{ac} = \frac{1}{\sqrt{2^L}} \Sigma_a X_a \exp\left(\frac{2\pi i}{2^L} ac\right)$$

In terms of the binary representations,

$$Y_c = \frac{1}{\sqrt{2^L}} \Sigma_a X_a \exp\left(\frac{2\pi i}{2^L} \Sigma_{j,k=0}^{L-1} a_j c_k 2^{j+k}\right)$$

Whenever $j + k \geq L$, we have $\omega^{(2^{j+k})} = 1$, so that we can drop those terms from consideration:

$$(FFT) \qquad Y_c = \frac{1}{\sqrt{2^L}} \Sigma_a X_a \exp\left(\frac{2\pi i}{2^L} \Sigma_{0 \leq j,k \leq L-1, j+k \leq L-1} a_j c_k 2^{j+k}\right)$$

Notice that all computations are in the field $\mathbf{Q}(\omega_{(2^L)})$.

Hadamard Transform

The Hadamard transform looks like a Fourier transform defined over $\mathbf{Z}_2^L$. It suits my purposes, pedagogically, to reverse the indexing on the output of the Hadamard transform, and get the transform

$$Y_c = \frac{1}{\sqrt{2^L}} \Sigma_a X_a (-1)^{(\Sigma_j a_j c_{L-1-j})}$$

(Normally the exponent would be $\Sigma_j a_j c_j$, but we reverse the indexing to bring out the similarity with the ordinary FFT.) Rewrite this as

$$Y_c = \frac{1}{\sqrt{2^L}} \Sigma_a X_a \exp\left(\frac{2\pi i}{2^L} \Sigma_{j=0}^{L-1} a_j c_{L-1-j} 2^{L-1}\right)$$

$$= \frac{1}{\sqrt{2^L}} \Sigma_a X_a \exp\left(\frac{2\pi i}{2^L} \Sigma_{0 \le j,k \le L-1; j+k=L-1} a_j c_k 2^{L-1}\right),$$

noting that $\exp(2\pi i 2^{L-1}/2^L) = -1$. Since the sum is restricted to those values of j, k satisfying $j + k = L - 1$, we can replace 2^{L-1} by 2^{j+k} and obtain

$$(HT) \qquad Y_c = \frac{1}{\sqrt{2^L}} \Sigma_a X_a \exp\left(\frac{2\pi i}{2^L} \Sigma_{0 \le j,k \le L-1; j+k=L-1} a_j c_k 2^{j+k}\right)$$

Approximate Fourier Transform

Comparing the two formulas (FFT) and (HT), we find that the only difference is in the limits on $j + k$: in (FFT) the range is $0 \le j + k \le L - 1$, while in (HT) the range is $L - 1 \le j + k \le L - 1$.

This leads us to define an Approximate Fourier Transform (AFFT), parameterized by an integer m :

$$(AFFT_m) \qquad Y_c = \frac{1}{\sqrt{2^L}} \Sigma_a X_a \exp\left(\frac{2\pi i}{2^L} \Sigma_{0 \le j,k \le L-1; L-m \le j+k \le L-1} a_j c_k 2^{j+k}\right)$$

When $m = 1$ this is the Hadamard transform (suitably indexed); when $m = L$ it becomes the ordinary Fourier transform.

Since $j + k \ge L - m$, the argument of "exp" is some multiple of $2\pi i 2^{L-m}/2^L = 2\pi i/2^m$, so that AFFT is defined over $\mathbf{Q}(\omega_{(2^m)})$.

The argument of "exp" in AFFT differs from that of FFT by

$$\frac{2\pi i}{2^L} \Sigma_{j+k<L-m} a_j c_k 2^{j+k}.$$

The magnitude of this difference is bounded by

$$\frac{2\pi}{2^L} L 2^{L-m} = 2\pi L 2^{-m}.$$

If $L = 500$ and $m = 20$, this bound is about 3/1000. So the matrix entries of AFFT differ from those of FFT by a multiplicative factor of $\exp(i\epsilon)$ where $|\epsilon| \leq 2\pi L 2^{-m} = 3/1000$. Thus if AFFT is used in place of FFT in Shor's factoring work [SHO], it leads to an overall error of a fraction of a degree in each phase angle, and less than one percent decrease in the magnitude of the probability of each desirable final state.

Calculating the AFFT

Start with the description of the Fast Fourier Transform as taken from [KNU, page 291, section 4.3.3]. I have replaced A, t, s, k by X, a, b, L, respectively, and numbered the passes from $L-1$ down to 0, to correspond to the bit being manipulated.

* Initialization. Let $X^{[L]}(a_{L-1}, ..., a_0) = X_a$, where $a = (a_{L-1}...a_0)_2$ (the binary representation).

* Pass J, $J = L-1, L-2, ..., 1, 0$. (Numbered downwards!) Set

* $X^{[J]}(b_{L-1}, b_{L-2}, ..., b_J, a_{J-1}, ..., a_0) :=$
$X^{[J+1]}(b_{L-1}, ..., b_{J+1}, 0, a_{J-1}, ..., a_0) +$
$\omega^{(b_J b_{J+1} ... b_{L-1} 0...0)_2} \times X^{[J+1]}(b_{L-1}, ..., b_{J+1}, 1, a_{J-1}, ..., a_0)$

We wish to compute the FFT quantum mechanically. At the outset, $X^{[L]}(a_{L-1}, ..., a_0)$ represents the amplitude of the state where L electrons have spins $a_{L-1}, ..., a_0$, respectively, with "1" representing "up" and "0" representing "down". Each succeeding $X^{[J]}(b_{L-1}, ..., a_0)$ represents the amplitude of the state of these same L electrons. The transform is performed by a sequence of two-electron interactions.

On Pass J, multiply the amplitudes $X^{[J+1]}(b_{L-1}, ..., b_{J+1}, 1, a_{J-1}, ..., a_0)$ (with a 1 in position J) by the phase shift $\omega^{(0 b_{J+1} ... b_{L-1} 0...0)_2}$. This correspond to the following two-bit operations. For each K, $J+1 \leq K \leq L-1$, use an interaction between electrons J and K to multiply the amplitude of those states with a 1 in both positions J and K by the factor

$$\omega^{(2^{L-1-K+J})}.$$

Call this transformation Q_{JK}.

Then apply the unitary transformation

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

to the electron J. Call this transformation P_J. So for $L = 3, J = 1$, the only value of K is $K = 2$, and we have

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & \omega^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \omega^0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \omega^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \omega^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \omega^2 \\ 0 & 0 & 0 & 0 & 1 & 0 & \omega^6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \omega^6 \end{bmatrix} = P_1 Q_{12} =$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 \end{bmatrix}$$

In general one would have $L - 1 - J$ of the two-bit interactions Q_{JK} on pass J, corresponding to different values of K. The entire 3-spin FFT is depicted in the Appendix.

So the FFT matrix is expressed as a product of unitary matrices. For example FFT on 4 electrons is

$$P_0 Q_{01} Q_{02} Q_{03} P_1 Q_{12} Q_{13} P_2 Q_{23} P_3.$$

If there are L electrons then there are L matrices P_J and $L(L-1)/2$ matrices Q_{JK}.

For our approximate AFFT, we simply delete those matrices Q_{JK} with $K \geq J + m$. So the AFFT is again unitary, and easily computed with one-bit and two-bit operators. It requires about Lm two-bit operations.

Quantum computation

Shor [SHO, page 12] suggests first developing a state

$$\frac{1}{\sqrt{q}} \Sigma_{a=0}^{q-1} |a>$$

where $q \approx 5n^2$ is a product of small prime powers, which will enable him to do a mixed-radix FFT later. By contrast, we suggest setting $q = 2^L \approx 5n^2$. Second, he computes $x^a (mod\ n)$, where x, n are integers computed classically, so that the state becomes

$$\frac{1}{\sqrt{q}} \Sigma_{a=0}^{q-1} |a, x^a>$$

Then he does the Fourier transform, sending a to c with amplitude $\frac{1}{\sqrt{q}} \exp(2\pi iac/q)$. This leaves the machine in state

$$\frac{1}{q} \Sigma_{a,c=0}^{q-1} \exp(2\pi iac/q)|c, x^a > .$$

We see that the radix-2^L Fourier transform is directly implementable as L^2 2-spin interactions, as opposed to the L^3 operations required by Shor.

We can improve still further, by doing our approximate Fourier transform instead of the Fourier transform. Notice that on Pass J of AFFT computation, we use interactions between bits J and K, $J < K < J + m$. So bit K of the output index, b_K, does not participate in any interaction after pass $J = K - m$. (Remember we are numbering backwards, so pass $K - m$ is m passes later than pass K.) Similarly, bit K of the input index, a_K, does not enter into the computation until pass $J = K$.

So we propose rearranging the computation in the following way.

```
* Start with $y = 1$ in an $L$-bit quantum register
      where you will compute $x^a$.
* For each $J = L-1, L-2, ..., 2, 1, 0$ :
  * Place the electron $J$ in state

        $\frac{1}{\sqrt{2}}(|0> + |1>)$

    corresponding to the two possible values of $a_J$.
  * Compute
        $y := y(x^{2^J})^{(a_J)}(mod\ n)$
    reversibly, in the register allocated for $y$.
  * For $K = J+1, J+2, ..., \min(J+m-1, L-1)$, apply operation $Q_{JK}$.
  * Apply operation $P_J$.
  * If $J \le L - m$, measure bit $b_{J+m-1} = c_{L-J-m}$ from the output
      of pass $J$ of the AFFT computation.
      (It will not enter any more interactions.)
* End (For each $J = L-1, L-2, ..., 2, 1, 0$).
* Measure the remaining bits $b_{m-2}, ..., b_0$.
```

* End algorithm

A possible advantage of this arrangement is that the electron in position K need only maintain coherence for m passes of the computation, although the rest of the system still has to maintain coherence for a longer time, so this advantage might be less than it appears at first blush.

A definite advantage is in the computational complexity. Shor's proposal, using a mixed-radix Fourier Transform with $q \approx 5n^2$ the product of small prime powers, appears to require about $(\log n)^3$ elementary operations (spin-spin interactions). The radix-2^L FFT requires only $(\log n)^2$ elementary operations. The AFFT requires only $(\log n)(\log\log n + \log 1/\epsilon)$ operations, where a final precision of ϵ is required. So the Fourier transform is no longer the bottleneck of the computation.

Parallel implementation

Several steps of the AFFT can be parallelized in the quantum implementation; this might further speed up the computation time, and increase the likelihood of the state remaining coherent until the computation is done.

We use, for an example, the FFT on 5 electrons, with operations proceeding right to left:

$$FFT = P_0 Q_{01} Q_{02} Q_{03} Q_{04} P_1 Q_{12} Q_{13} Q_{14} P_2 Q_{23} Q_{24} P_3 Q_{34} P_4$$

We can interchange the order of any two operations which do not involve any of the same electrons; alternatively, we can do such operations in parallel. At time step $K = 8, 7, ..., 1, 0$, let us perform P_I if $I + I = K$, and Q_{IJ} if $I + J = K$. Steps that are performed in parallel are displayed within square brackets, vertically aligned, and again proceeding right to left:

$$FFT = P_0 Q_{01} \begin{bmatrix} P_1 \\ Q_{02} \end{bmatrix} \begin{bmatrix} Q_{12} \\ Q_{03} \end{bmatrix} \begin{bmatrix} P_2 \\ Q_{13} \\ Q_{04} \end{bmatrix} \begin{bmatrix} Q_{23} \\ Q_{14} \end{bmatrix} \begin{bmatrix} P_3 \\ Q_{24} \end{bmatrix} Q_{34} P_4$$

We used 9 time steps here; for an L-electron system we will use $2L-1$ time steps.

This parallel implementation looks a lot like "systolic arrays," [MC, chapter 8, section 8.3], and suggests directions for physical implementation.

References

[KNU] Donald E. Knuth, volume 2. The Art of Computer Programming, Volume 2: Seminumerical Algorithms. (Addison-Wesley, Reading, MA, 2nd ed., 1981)

[MC] Carver Mead and Lynn Conway, Introduction to VLSI Systems. (Addison-Wesley, Reading, MA, 1980)

[SHO] Peter W. Shor, “Algorithms for Quantum Computation: Discrete Log and Factoring,” manuscript, 1994. Proceedings of FOCS 1994.

Appendix

We write out in full the FFT on 3 electrons. Note that the rows are numbered in bit-reversed order (04261537), corresponding to the index of b rather than c.

$$FFT = P_0 Q_{01} Q_{02} P_1 Q_{12} P_2 =$$

$$= \frac{1}{\sqrt{2}} \underset{P_0}{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}} \times \underset{Q_{01}}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 \end{bmatrix}} \times$$

$$\times \underset{Q_{02}}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega \end{bmatrix}} \times \frac{1}{\sqrt{2}} \underset{P_1}{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}} \times$$

$$\times \underset{Q_{12}}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 \end{bmatrix}} \times \frac{1}{\sqrt{2}} \underset{P_2}{\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}}$$

$$= \frac{1}{\sqrt{8}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^0 & \omega^2 & \omega^4 & \omega^6 \\ \omega^0 & \omega^6 & \omega^4 & \omega^2 & \omega^0 & \omega^6 & \omega^4 & \omega^2 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega^1 & \omega^6 & \omega^3 \\ \omega^0 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ \omega^0 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{bmatrix}$$