Quantum Logic Gates and Circuits

Physical and Logical Reversibility

Classical and Quantum Logic

- Classical Logic
  - Typically is Irreversible Logic
  - Reversible Logic is a special case
  - Fanout is a powerful use-case
- Quantum Logic
  - Comparison with Classical
  - No Cloning theorem
  - Universal Quantum Logic Gates

Classical Irreversible Logic

- Theory of Nature of Computing (Church, Turing 1936)
- Universality of Primitive Operations

x	*y*	*x* AND *y*
0	0	0
0	1	0
1	0	0
1	1	1

x	*y*	*x* OR *y*
0	0	0
0	1	1
1	0	1
1	1	1

y	NOT *y*
0	1
1	0

Types of Reversibility

- Logical Reversibility
  - Ability to reconstruct input from output. Circuit function is a *Bijection*.
    - Bijection implies two properties: (1) one-to-one, (2) onto
- Physical Reversibility
  - Thermodynamic entropy based arguments that relate the loss of information to an increase in dissipated heat.
  - Heat dissipation during a computation is generally a sign of physical irreversibility.

Thermodynamics Concepts (Oversimplified)

- Thermodynamics
  - branch of physics that studies the effects of changes in temperature, pressure, and volume in physical systems
- Physical System
  - A region of spacetime and all entities (particles and fields) contained within it. (eg. universe, transistors, circuits, computers - defn from M. Frank)
- Entropy
  - measure of the amount of energy in a physical system that cannot be used to do work - entropy S is multiplied by a temperature to yield an amount of energy. It is a measure of the disorder and randomness present in a system. A quantitative measure of the amount of thermal energy NOT AVAILABLE to do work.

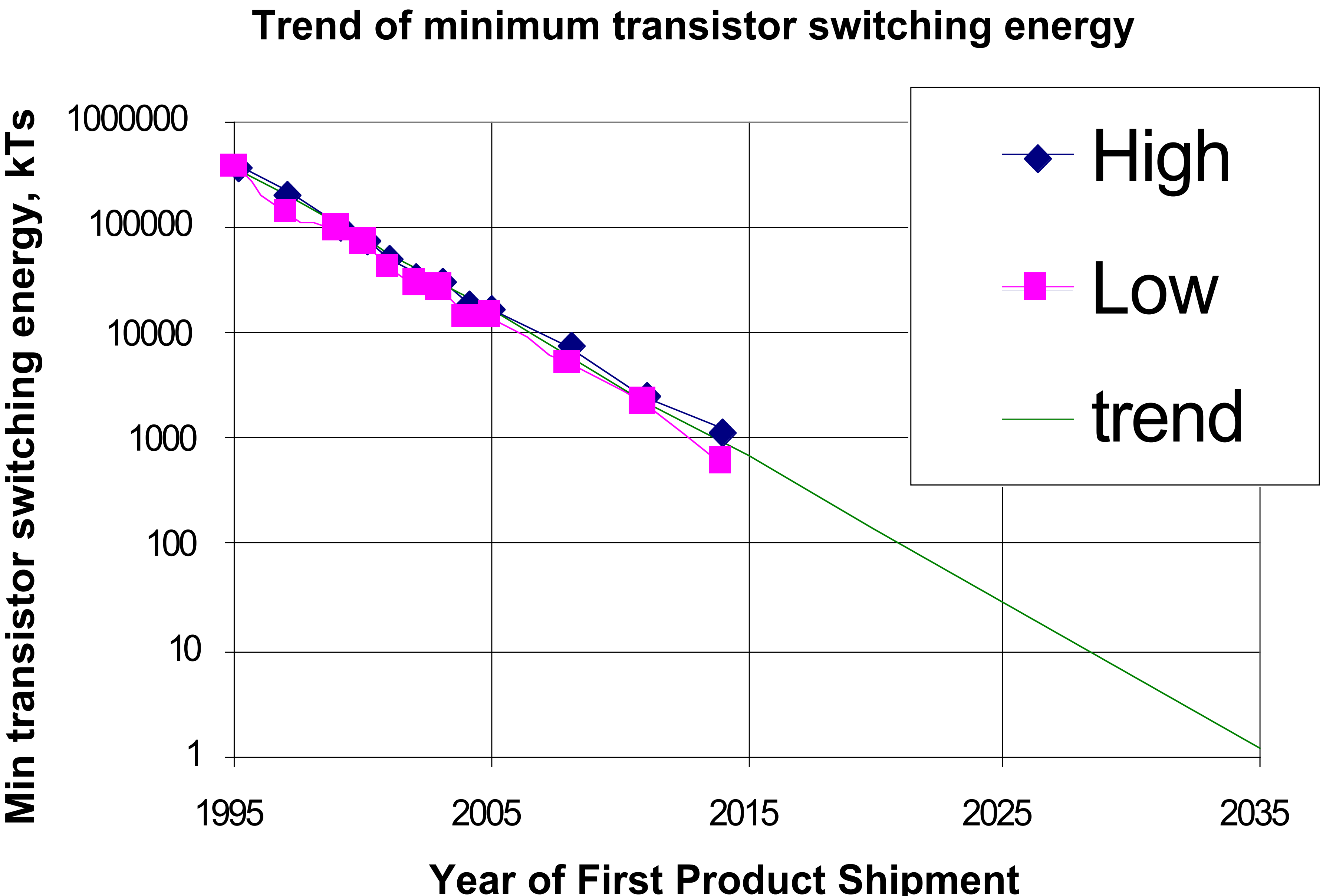
Physical Irreversibility

- 2-input NAND Gate
  - one output, two inputs
  - in computing an output, one input is "erased"
  - information irretrievably lost
  - change in entropy of the system is one bit of information - quantitatively this is $ln\ 2$
  - conversion to energy increase of $kT\ ln\ 2$ where k is Boltzman's constant and T is temperature
  - corresponds to energy "lost" to heat dissipation and a sign of physical irreversibility

Developments in Reversibility

- Can a computation be accomplished in a logically reversible fashion? (unlike using a NAND gate - 1970's)
- Must heat be dissipated during a computation?
  - Feynmann points out (1986) transistor dissipates $10^{10} kT$ joules of heat, DNA copying in a human cell dissipates $100\ kT$ joules all far from $0.693\ kT$ joule lower bound from erasing a single bit

Min. Transistor Switching Energy Trend*



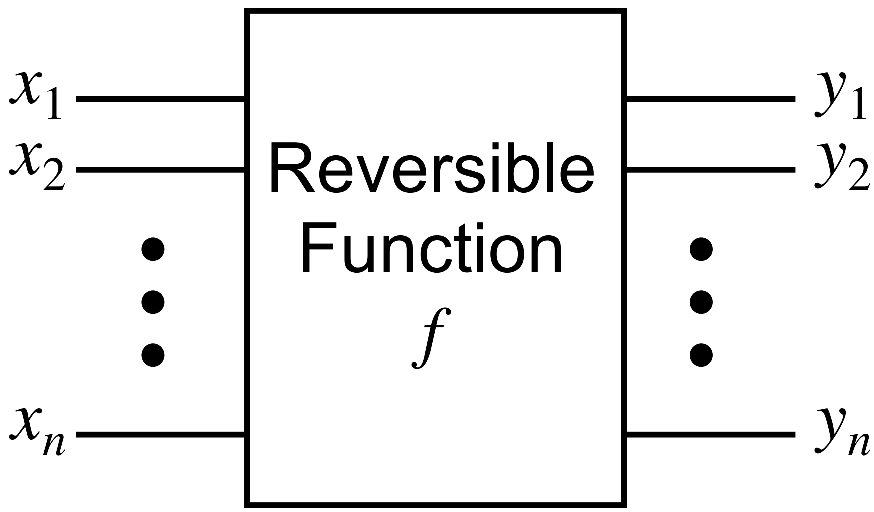
½CV^2 based on ITRS '99 figures for V_{dd} and minimum transistor gate capacitance. T=300 K

*based on chart prepared by M. Frank at Univ. of Fla.

Developments in Reversibility

- 1973 - Bennett proved that classical computation can be accomplished with no energy dissipated per computational step and with reversibility (reversible Turing machine model)
- This triggered a search for physical models for reversible classical computation
- Common Model is a discrete one-to-one binary-valued Boolean function with an equal number of inputs and outputs

Reversible Logic Circuit



- f is a bijective function
- contains symmetry that allows for other forms of representation (transformation matrix)

Quantum Logic Gates and Circuits

Single Qubit "Gates" or Transformation Operators

Quantum Logic Gates and Circuits

- Quantum Gates: Building Blocks of Quantum Computers
- Quantum Gate transforms a Quantum State to a New State
- State Transformations Performed by Gates Described by Hermitian Operators
- Matrix Describing State Transformation is Transfer Matrix

Matrices in Quantum Mechanics

- Normal Matrices, $\mathbf{N}$:

$$\mathbf{N}\mathbf{N}^{\dagger} = \mathbf{N}^{\dagger}\mathbf{N}$$

- Hermitian Matrices, $\mathbf{H}$:

$$\mathbf{H}\mathbf{H}^{\dagger} = \mathbf{H}^{\dagger}\mathbf{H}$$

$$\mathbf{H} = \mathbf{H}^{\dagger}$$

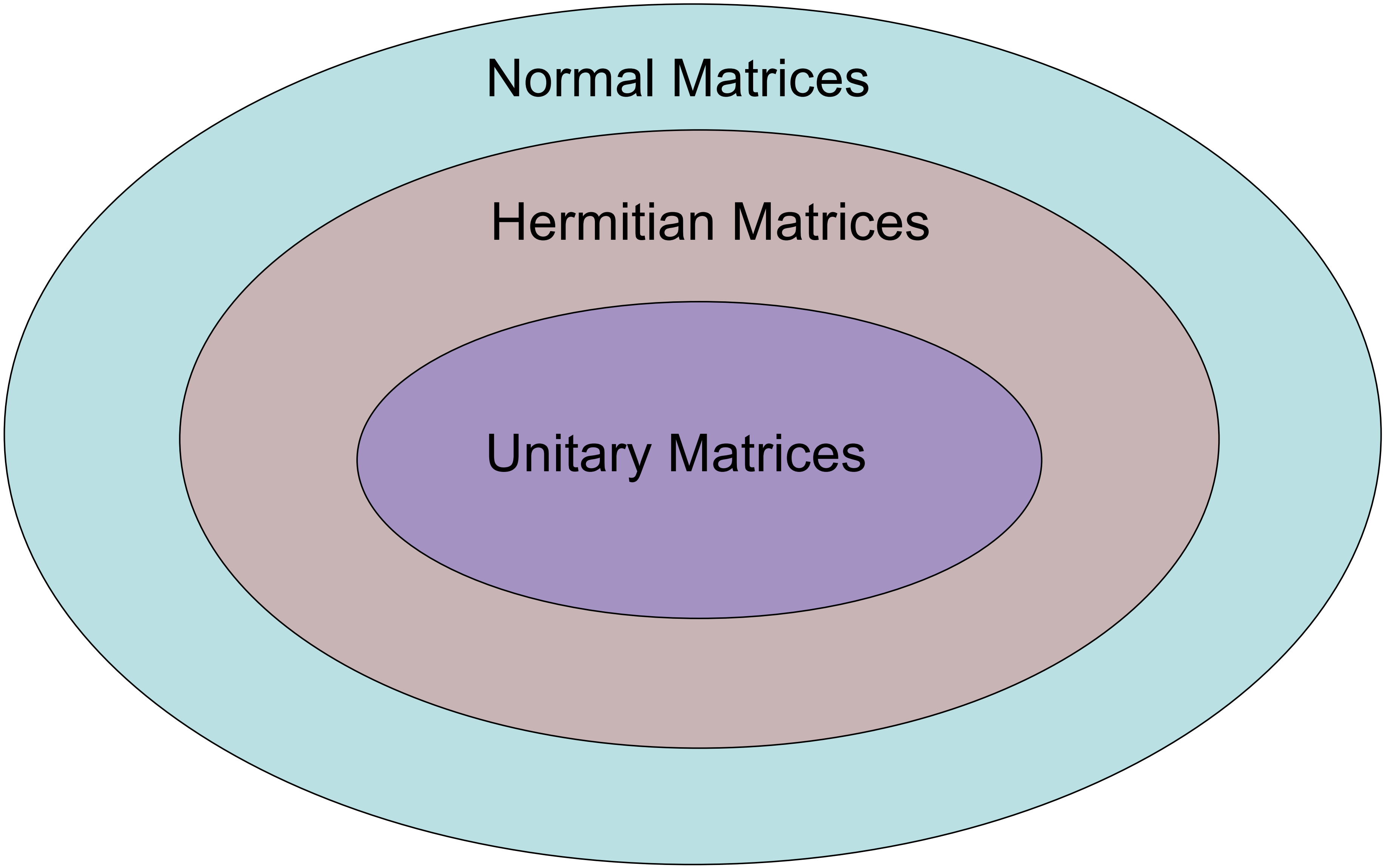
- Unitary Matrices, $\mathbf{U}$:

$$\mathbf{U}\mathbf{U}^{\dagger} = \mathbf{U}^{\dagger}\mathbf{U}$$

$$\mathbf{U} = \mathbf{U}^{\dagger}$$

$$\mathbf{U}^{-1} = \mathbf{U}^{\dagger}$$

Matrices in Quantum Mechanics



Practice Problems

- Given that if a square matrix $\mathbf{A}$ satisfies $\mathbf{A}^{-1}=\mathbf{A}^{\dagger}$, then it must also be Hermitian (*i.e.*, that $\mathbf{A}=\mathbf{A}^{\dagger}$).
- Prove that the eigenvalues of a Hermitian matrix are always real-valued.
- Prove that the eigenvalues of a Unitary matrix always have unity magnitude (*i.e.*, their norm is always one).
- Prove that the non-trivial eigenvectors of a Hermitian matrix are always orthogonal with one another.

Quantum Logic Gate Matrices

- Matrices are Unitary

$$\mathbf{U}\mathbf{U}^{\dagger} = \mathbf{U}^{\dagger}\mathbf{U} = \mathbf{I}$$

$$\mathbf{U}^{-1} = \mathbf{U}^{\dagger}$$

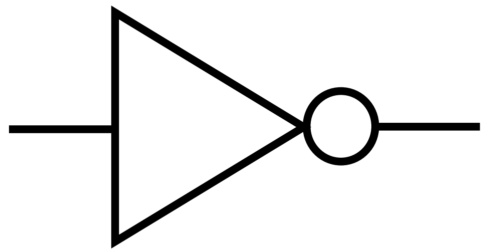
- Transformations are Reversible
- Unitary Transformations Corresponds to:
  - Length Preservation
  - Information Preserving Rotation in Vector Space

Quantum Logic Gate Matrices

- Gates have equal number of Inputs and Outputs
- Input/Output States of Quantum Gate Described by Vectors in Hilbert Space
- 1-qubit: $\mathbb{H}^2$ $\{|0\rangle, |1\rangle\}$
- 2-qubit: $\mathbb{H}^4 = \mathbb{H}^2 \otimes \mathbb{H}^2$ $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$
- 3-qubit: $\mathbb{H}^8 = \mathbb{H}^2 \otimes \mathbb{H}^2 \otimes \mathbb{H}^2$

$$\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$$

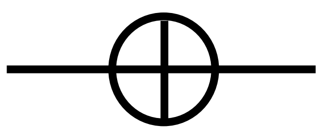
Classical Reversible Gates/Operators



Classic Symbol
Physically Irreversible
Logically Reversible

In	Out
0	1
1	0

Classic Truth Table
Notion of Inputs/Outputs

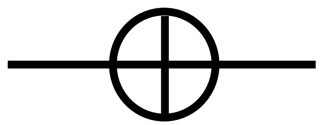


NOT
Symbol for Reversible
NOT Gate
(aka Pauli-$\mathbf{X}$)

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Matrix Representation
of Pauli-$\mathbf{X}$ Gate
Functionality

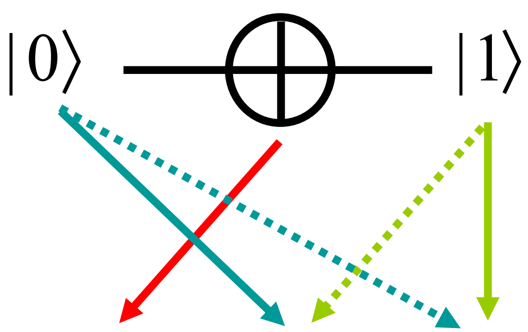
Reversible NOT Gate



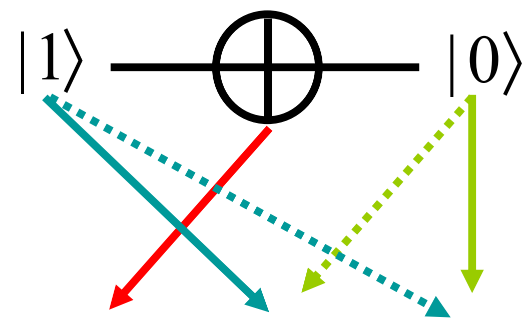
NOT
Symbol for Reversible NOT Gate

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Matrix Representation of NOT Gate Functionality

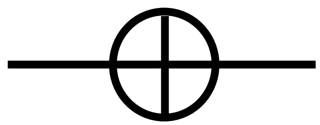


$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Reversible NOT Gate



NOT

Symbol for Reversible NOT Gate

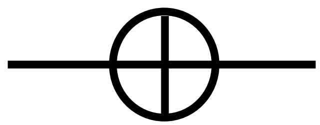
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Matrix Representation of NOT Gate Functionality

$$\mathbf{A}^{\dagger} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{A}^{\dagger}\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Reversible NOT Gate



NOT

Symbol for Reversible NOT Gate

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

Matrix Representation of NOT Gate Functionality

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

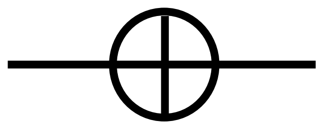
Dirac Notation Example

$$\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)|0\rangle$$

$$= |0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle$$

$$= |0\rangle(0) + |1\rangle(1) = |1\rangle$$

Reversible NOT Gate



NOT

Symbol for Reversible NOT Gate

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

Matrix Representation of NOT Gate Functionality

"INPUTS"

"OUTPUTS"

$$\begin{array}{c|cc} & |0\rangle & |1\rangle \\ \hline |0\rangle & 0 & 1 \\ |1\rangle & 1 & 0 \end{array}$$

Permutation Matrix Transformations of Qubits

Derivation of $\mathbf{I}_2$ or σ_0

- This Operator Performs an Identity Transformation of the Basis Vectors:

$$|0\rangle \mapsto |0\rangle \qquad\qquad |1\rangle \mapsto |1\rangle$$

- Computed as:

$$\sigma_0 = \mathbf{I} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$\sigma_0 = \mathbf{I} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\sigma_0 = \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Derivation of $\mathbf{X}$ or σ_X

- This Operator "Flips" or "Negates" a Qubit:

$$|0\rangle \mapsto |1\rangle \qquad |1\rangle \mapsto |0\rangle$$

- Computed as:

$$\sigma_1 = \mathbf{X} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$\sigma_1 = \mathbf{X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\sigma_1 = \mathbf{X} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Derivation of $\mathbf{Y}$ or σ_Y

- This Operator Multiplies a Qubit by i (90-degree phase shift) then "Flips" or "Negates" it:

$$|0\rangle \mapsto i\,|1\rangle \qquad\qquad |1\rangle \mapsto -i\,|0\rangle$$

- Computed as:

$$\sigma_2 = \mathbf{Y} = -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|$$

$$\sigma_2 = \mathbf{Y} = -i\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0 & 1\end{bmatrix} + i\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1 & 0\end{bmatrix}$$

$$\sigma_2 = \mathbf{Y} = -i\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + i\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$$

Derivation of $\mathbf{Z}$ or σ_Z

- This Operator is an Identity with a 180 Degree Phase Shift Operation:

$$|0\rangle \mapsto |0\rangle \qquad\qquad |1\rangle \mapsto -|1\rangle$$

- Computed as:

$$\sigma_3 = \mathbf{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\sigma_3 = \mathbf{Z} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\sigma_3 = \mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Pauli Operator Examples

- Assume the Following:

$$|\varphi\rangle = \sigma_i |\psi\rangle = \sigma_i[\alpha_0 |0\rangle + \alpha_1 |1\rangle]$$

$$|\varphi\rangle = \sigma_0 |\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

$$|\varphi\rangle = \sigma_1 |\psi\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_0 \end{bmatrix} = \alpha_1 |0\rangle + \alpha_0 |1\rangle$$

$$|\varphi\rangle = \sigma_2 |\psi\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = i \begin{bmatrix} -\alpha_1 \\ \alpha_0 \end{bmatrix} = -i\alpha_1 |0\rangle + i\alpha_0 |1\rangle$$

$$|\varphi\rangle = \sigma_3 |\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ -\alpha_1 \end{bmatrix} = \alpha_0 |0\rangle - \alpha_1 |1\rangle$$

Quantum Logic Gates and Circuits

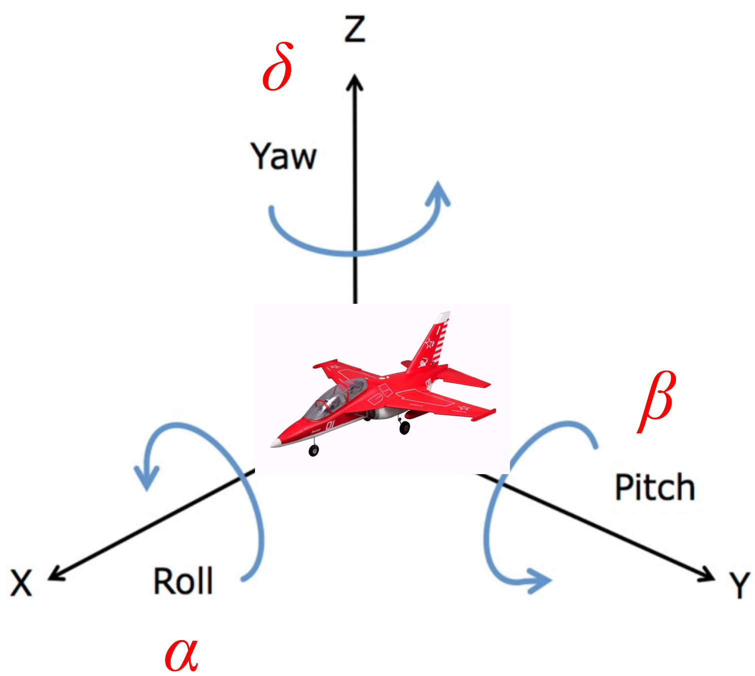
Single Qubit "Gates" or Transformation Operators

GEOMETRIC INTERPRETATIONS

Orientation and Definitions from Aerospace

- 3-D Rotation Matrices are Elements of the non-Abelian $\mathrm{SO}(3)$ Group with Direct Matrix Multiplication as the Group Product Operation

$$\mathbf{R}(\omega) = \mathbf{R}_{\mathbf{X}}(\alpha)\mathbf{R}_{\mathbf{Y}}(\beta)\mathbf{R}_{\mathbf{Z}}(\delta)$$

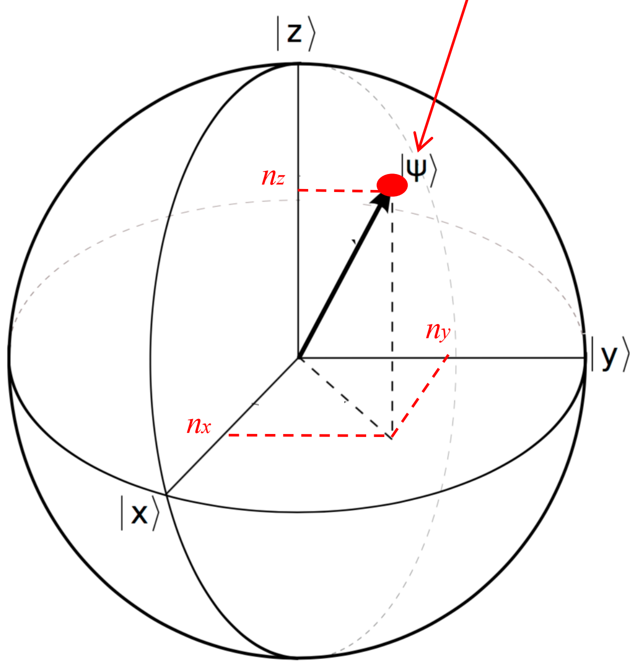


Qubit on the Bloch Sphere Surface (Cartesian or rectangular)

Cartesian Bloch Vector:

$$\hat{\mathbf{n}}_{|\psi_0\rangle} = (n_x, n_y, n_z)^{\mathrm{T}}$$

$|\psi_0\rangle$



- Location on Surface of Sphere in Rectangular Coordinates:

$$|\psi_0\rangle = n_x|x\rangle + n_y|y\rangle + n_z|z\rangle$$

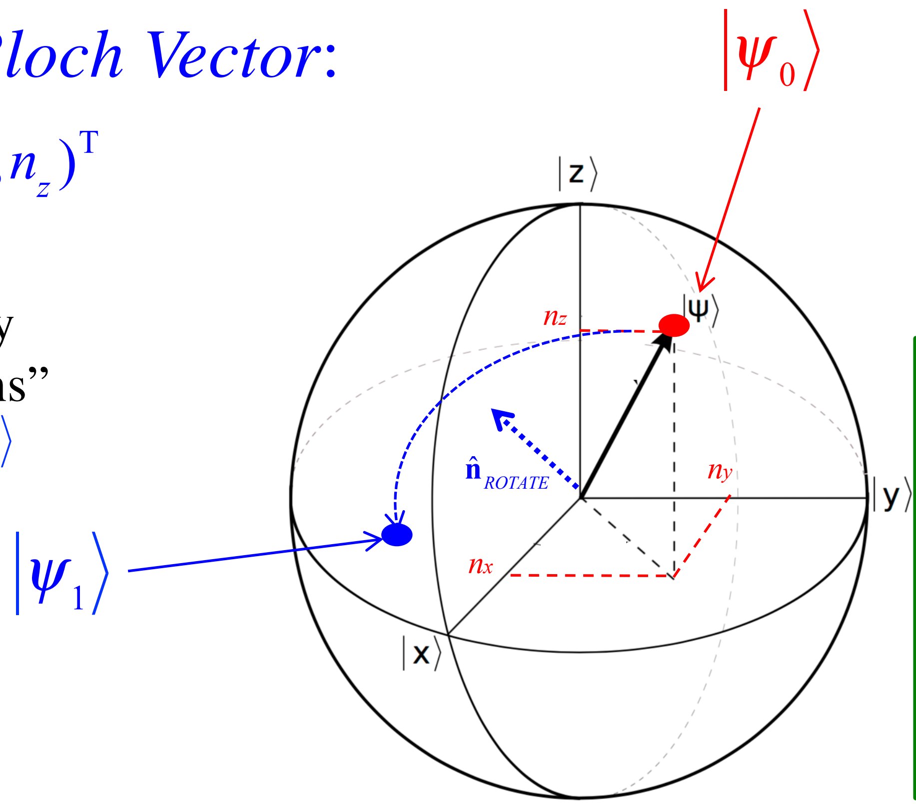
- <u>Non-traditional</u> Basis, $\left(|x\rangle, |y\rangle, |z\rangle\right)$

Qubit Rotation on the Bloch Sphere Surface (Cartesian or rectangular)

Cartesian Bloch Vector:

$$\hat{\mathbf{n}}_{|\psi_0\rangle} = (n_x, n_y, n_z)^{\mathrm{T}}$$

There are many Different "Paths" from $|\psi_0\rangle$ to $|\psi_1\rangle$



$$|\psi_0\rangle \rightarrow |\psi_1\rangle$$

$$|\psi_1\rangle = \mathbf{R}|\psi_0\rangle$$

- Geodesic or "Great Circle" Path is Shown with Blue Dashed Line
- Shortest Path Rotation
- Great Circle Rotation is Rotation of Angle ω about an Axis of Rotation: $\hat{\mathbf{n}}_{ROTATE}$

- **R** is Generally Defined about a General "Axis of Rotation"
  - $\hat{\mathbf{n}}_{ROTATE} = \hat{\mathbf{n}}_{|\psi_0\rangle} \times \hat{\mathbf{n}}_{|\psi_1\rangle}$ (3D Vector Cross Product, Right-hand Rule)
- **R** is a Single <u>3 × 3</u> Rotation Matrix – <u>Unconventional</u> in QIS
  - $\mathbf{R}\left(\hat{\mathbf{n}}_{ROTATE}, \omega\right)$ (Explicit Formula in Backup Slides, Simpler using Quaternions)

Rotation Operator, $\mathbf{R}$, Product of Elemental Rotations

- $\mathbf{R}$ can be Defined as a Product of Rotations about Three Axes, (x,y,z):

$$|\psi_0\rangle \rightarrow |\psi_1\rangle$$

$$|\psi_1\rangle = \mathbf{R}|\psi_0\rangle$$

- Two forms of Elemental Rotation Decompositions
  - Product of 3 Rotation Matrices about Each Axis: Tait-Bryan Rotations
  - Product of 3 Rotation Matrices about Two of the Axes: Euler Rotations
- Tait-Bryan Forms:

$$\mathbf{R} = \mathbf{R}_{x}(\alpha)\mathbf{R}_{y}(\beta)\mathbf{R}_{z}(\delta), \mathbf{R} = \mathbf{R}_{y}(\beta)\mathbf{R}_{z}(\delta)\mathbf{R}_{x}(\alpha), \mathbf{R} = \mathbf{R}_{z}(\delta)\mathbf{R}_{x}(\alpha)\mathbf{R}_{y}(\beta),$$

$$\mathbf{R} = \mathbf{R}_{x}(\alpha)\mathbf{R}_{z}(\delta)\mathbf{R}_{y}(\beta), \mathbf{R} = \mathbf{R}_{z}(\delta)\mathbf{R}_{y}(\beta)\mathbf{R}_{x}(\alpha), \mathbf{R} = \mathbf{R}_{y}(\beta)\mathbf{R}_{x}(\alpha)\mathbf{R}_{z}(\delta)$$

- Euler Forms:

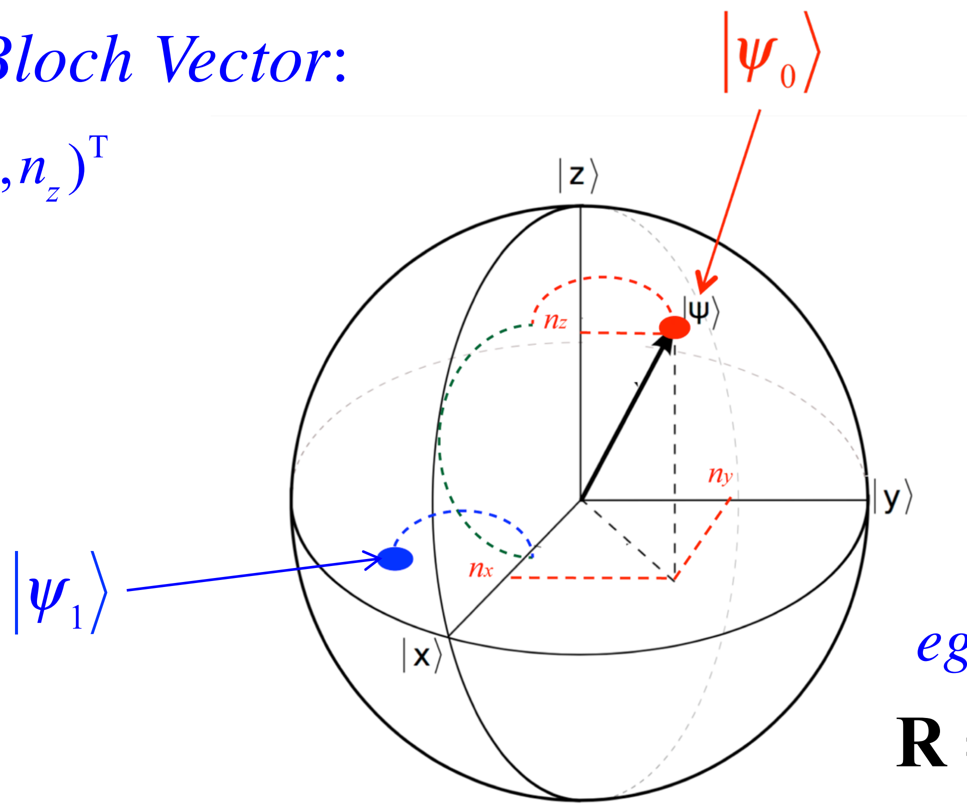
$$\mathbf{R} = \mathbf{R}_{z}(\delta_1)\mathbf{R}_{x}(\alpha)\mathbf{R}_{z}(\delta_2), \mathbf{R} = \mathbf{R}_{x}(\alpha_1)\mathbf{R}_{y}(\beta)\mathbf{R}_{x}(\alpha_2), \mathbf{R} = \mathbf{R}_{y}(\beta_1)\mathbf{R}_{z}(\delta)\mathbf{R}_{y}(\beta_2),$$

$$\mathbf{R} = \mathbf{R}_{z}(\delta_1)\mathbf{R}_{y}(\beta)\mathbf{R}_{z}(\delta_2), \mathbf{R} = \mathbf{R}_{x}(\alpha_1)\mathbf{R}_{z}(\delta)\mathbf{R}_{x}(\alpha_2), \mathbf{R} = \mathbf{R}_{y}(\beta_1)\mathbf{R}_{x}(\alpha)\mathbf{R}_{y}(\beta_2)$$

Qubit Rotation on the Bloch Sphere Surface (Elemental Rotation)

Cartesian Bloch Vector:

$\hat{\mathbf{n}}_{|\psi_0\rangle} = (n_x, n_y, n_z)^T$



$|\psi_0\rangle \rightarrow |\psi_1\rangle$

$|\psi_1\rangle = \mathbf{R}|\psi_0\rangle$

eg., Tait-Bryan form: (z-y-x)

$$\mathbf{R} = \mathbf{R}_X(\alpha)\mathbf{R}_Y(\beta)\mathbf{R}_Z(\delta)$$

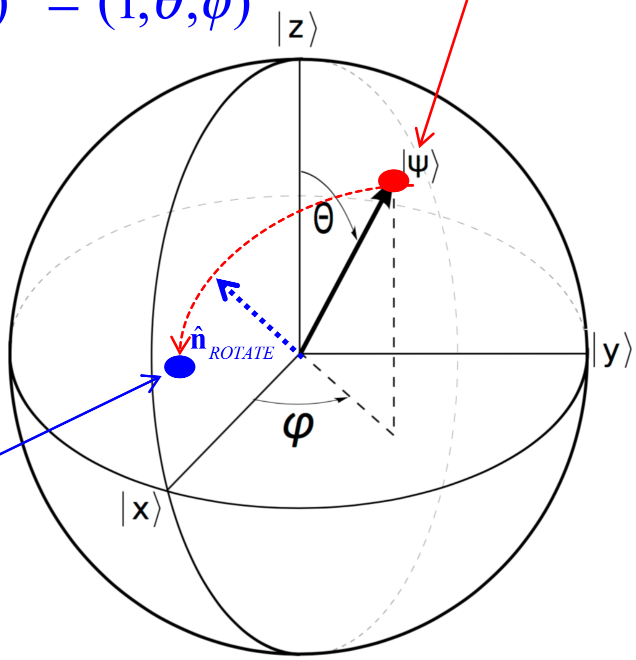
$$\mathbf{R}_X(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \quad \mathbf{R}_Y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \quad \mathbf{R}_Z(\delta) = \begin{bmatrix} \cos\delta & -\sin\delta & 0 \\ \sin\delta & \cos\delta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Moving on the Bloch Sphere Surface (Spherical)

Bloch Vector = (r, φ, θ)

$\hat{\mathbf{n}}_{|\psi_0\rangle} = (r,\theta,\phi)^T = (1,\theta,\phi)^T$

$|\psi_0\rangle$



$|\psi_1\rangle = \mathbf{U}|\psi_0\rangle$

$$r = \left\| \psi_0 \right\| = \langle \psi_0 | \psi_0 \rangle = \sqrt{n_x^2 + n_y^2 + n_z^2} = 1$$

$$\varphi = \tan^{-1}\left(\frac{n_y}{n_x}\right) \qquad \theta = \cos^{-1}\left(\frac{n_z}{\sqrt{n_x^2 + n_y^2 + n_z^2}}\right)$$

$$n_x = \langle \psi_0 | \psi_0 \rangle \sin\theta\cos\varphi = \sin\theta\cos\varphi$$

$$n_y = \langle \psi_0 | \psi_0 \rangle \sin\theta\sin\varphi = \sin\theta\sin\varphi$$

$$n_z = \langle \psi_0 | \psi_0 \rangle \cos\theta = \cos\theta$$

- Location on Surface of Sphere in Rectangular Coordinates:

$$|\psi_0\rangle = n_x|x\rangle + n_y|y\rangle + n_z|z\rangle = (\sin\theta\cos\varphi)|x\rangle + (\sin\theta\sin\varphi)|y\rangle + (\cos\theta)|z\rangle$$

$$= e^{i\gamma}\left[\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right]$$

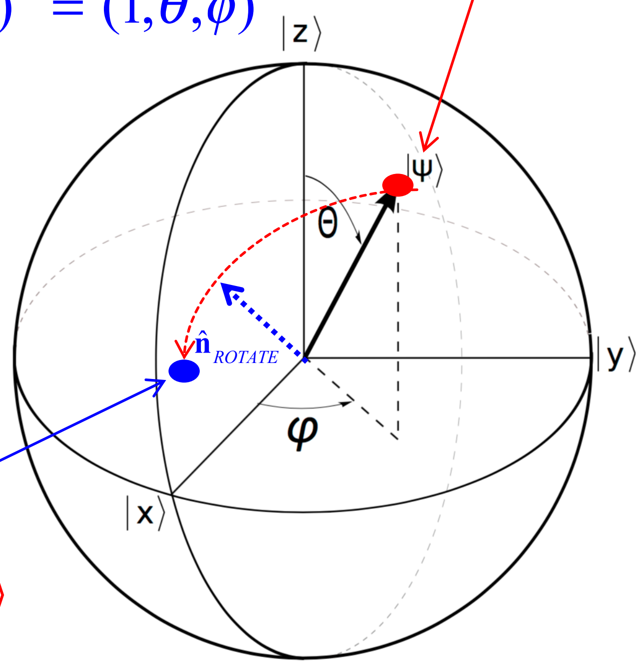
Global Phase γ is not Important in Terms of Information Content

Moving on the Bloch Sphere Surface (Spherical)

Bloch Vector $= (r, \varphi, \theta)$

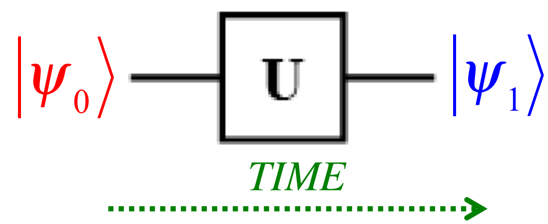
$$\hat{\mathbf{n}}_{|\psi_0\rangle} = (r, \theta, \phi)^{\mathrm{T}} = (1, \theta, \phi)^{\mathrm{T}}$$

$|\psi_0\rangle$



$$|\psi_1\rangle = \mathbf{U}|\psi_0\rangle$$

$|\psi_0\rangle$ Evolves in Time to $|\psi_1\rangle$



$$\mathbf{U}_{3D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & u_{00} & u_{01} \\ 0 & u_{10} & u_{11} \end{bmatrix} = \mathbf{R_X}(\alpha)\mathbf{R_Y}(\beta)\mathbf{R_Z}(\delta)$$

Euler or Tait-Bryan Decompositions

$$\mathbf{R_X}(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\alpha}{2} & -i\sin\frac{\alpha}{2} \\ 0 & -i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix}$$

$$\mathbf{R_Y}(\beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ 0 & \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix}$$

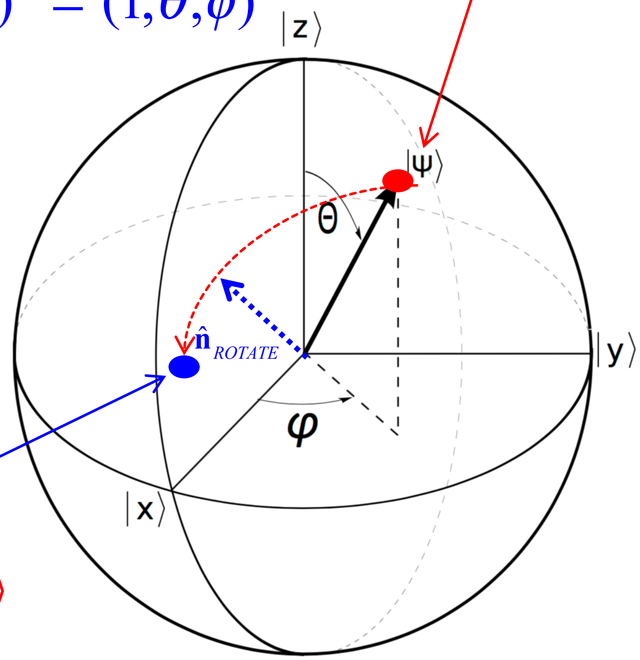
$$\mathbf{R_Z}(\delta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-i\frac{\delta}{2}} & 0 \\ 0 & 0 & e^{i\frac{\delta}{2}} \end{bmatrix}$$

Moving on the Bloch Sphere Surface (Spherical)

Bloch Vector $= (r, \varphi, \theta)$

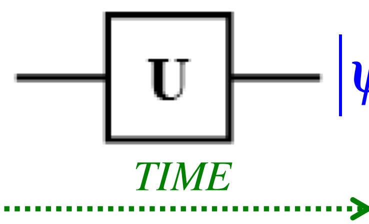
$$\hat{\mathbf{n}}_{|\psi_0\rangle} = (r,\theta,\phi)^T = (1,\theta,\phi)^T$$

$|\psi_0\rangle$



$$|\psi_1\rangle = \mathbf{U}|\psi_0\rangle$$

$|\psi_0\rangle$ Evolves in Time to $|\psi_1\rangle$



$$\mathbf{U} = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix} = \mathbf{R_X}(\alpha)\mathbf{R_Y}(\beta)\mathbf{R_Z}(\delta)$$

Euler or Tait-Bryan Decompositions

$$\mathbf{R_X}(\alpha) = \begin{bmatrix} \cos\frac{\alpha}{2} & -i\sin\frac{\alpha}{2} \\ -i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix} \qquad \mathbf{R_Y}(\beta) = \begin{bmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix} \qquad \mathbf{R_Z}(\delta) = \begin{bmatrix} e^{-i\frac{\delta}{2}} & 0 \\ 0 & e^{i\frac{\delta}{2}} \end{bmatrix}$$

Rotation Identities (Euler Decompositions)

Rotate γ Degrees about **X**-axis	Rotate γ Degrees about **Y**-axis	Rotate γ Degrees about **Z**-axis
$\mathbf{R}_X(\gamma)=\mathbf{R}_Z\left(-\frac{\pi}{2}\right)\mathbf{R}_Y(\gamma)\mathbf{R}_Z\left(\frac{\pi}{2}\right)$	$\mathbf{R}_Y(\gamma)=\mathbf{R}_X\left(-\frac{\pi}{2}\right)\mathbf{R}_Z(\gamma)\mathbf{R}_X\left(\frac{\pi}{2}\right)$	$\mathbf{R}_Z(\gamma)=\mathbf{R}_X\left(\frac{\pi}{2}\right)\mathbf{R}_Y(\gamma)\mathbf{R}_X\left(-\frac{\pi}{2}\right)$
$\mathbf{R}_X(\gamma)=\mathbf{R}_Y\left(\frac{\pi}{2}\right)\mathbf{R}_Z(\gamma)\mathbf{R}_Y\left(-\frac{\pi}{2}\right)$	$\mathbf{R}_Y(\gamma)=\mathbf{R}_Z\left(\frac{\pi}{2}\right)\mathbf{R}_X(\gamma)\mathbf{R}_Z\left(-\frac{\pi}{2}\right)$	$\mathbf{R}_Z(\gamma)=\mathbf{R}_Y\left(-\frac{\pi}{2}\right)\mathbf{R}_X(\gamma)\mathbf{R}_Y\left(\frac{\pi}{2}\right)$

- Notes:
  - These Examples Restrict Decomposed Euler Angles to $(\pm(\pi/2), \gamma, \pm(\pi/2))$
  - Angles are Restricted to the Interval $[-\pi, +\pi]$
  - Rotations of $\pm 2m\pi$ for m=0,1,2,... are Omitted, $\mathbf{R}_k(\pm 2m\pi)=\mathbf{R}_k(0)=\mathbf{I}$

Rotation Identities (Tait-Bryan Decompositions)

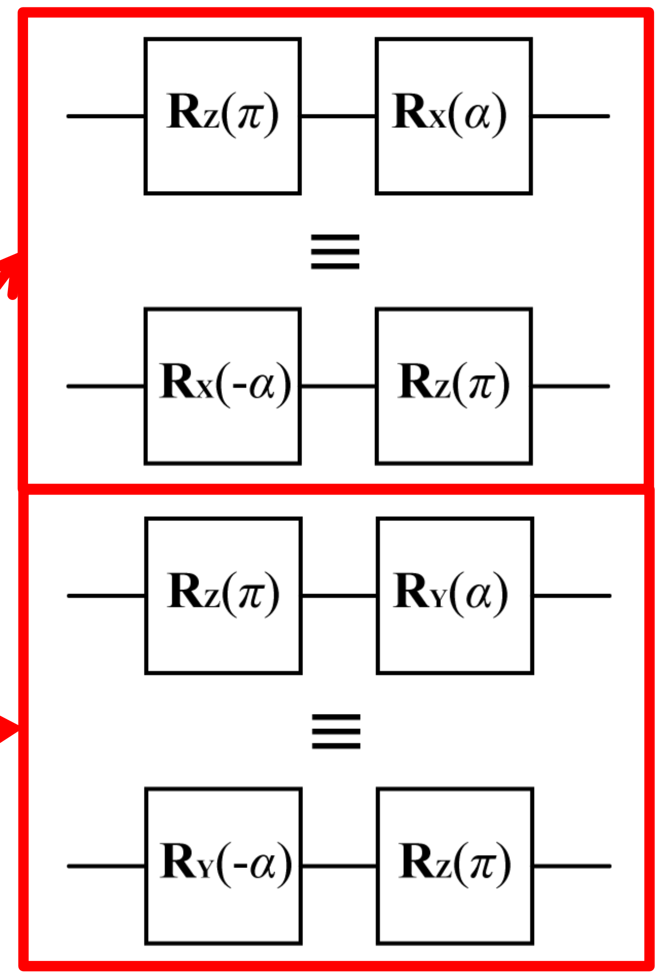
Rotate γ Degrees about **X**-axis	Rotate γ Degrees about **Y**-axis	Rotate γ Degrees about **Z**-axis
$\mathbf{R}_X(\gamma)=\mathbf{R}_X(\alpha)\mathbf{R}_Y(\beta)\mathbf{R}_Z(\delta)$	$\mathbf{R}_Y(\gamma)=\mathbf{R}_X(\alpha)\mathbf{R}_Y(\beta)\mathbf{R}_Z(\delta)$	$\mathbf{R}_Z(\gamma)=\mathbf{R}_X(\alpha)\mathbf{R}_Y(\beta)\mathbf{R}_Z(\delta)$
$\mathbf{R}_X(\gamma)=\mathbf{R}_Y(\beta)\mathbf{R}_Z(\delta)\mathbf{R}_X(\alpha)$	$\mathbf{R}_Y(\gamma)=\mathbf{R}_Y(\beta)\mathbf{R}_Z(\delta)\mathbf{R}_X(\alpha)$	$\mathbf{R}_Z(\gamma)=\mathbf{R}_Y(\beta)\mathbf{R}_Z(\delta)\mathbf{R}_X(\alpha)$
$\mathbf{R}_X(\gamma)=\mathbf{R}_Z(\delta)\mathbf{R}_X(\alpha)\mathbf{R}_Y(\beta)$	$\mathbf{R}_Y(\gamma)=\mathbf{R}_Z(\delta)\mathbf{R}_X(\alpha)\mathbf{R}_Y(\beta)$	$\mathbf{R}_Z(\gamma)=\mathbf{R}_Z(\delta)\mathbf{R}_X(\alpha)\mathbf{R}_Y(\beta)$
$\mathbf{R}_X(\gamma)=\mathbf{R}_X(\alpha)\mathbf{R}_Z(\delta)\mathbf{R}_Y(\beta)$	$\mathbf{R}_Y(\gamma)=\mathbf{R}_X(\alpha)\mathbf{R}_Z(\delta)\mathbf{R}_Y(\beta)$	$\mathbf{R}_Z(\gamma)=\mathbf{R}_X(\alpha)\mathbf{R}_Z(\delta)\mathbf{R}_Y(\beta)$
$\mathbf{R}_X(\gamma)=\mathbf{R}_Z(\delta)\mathbf{R}_Y(\beta)\mathbf{R}_X(\alpha)$	$\mathbf{R}_Y(\gamma)=\mathbf{R}_Z(\delta)\mathbf{R}_Y(\beta)\mathbf{R}_X(\alpha)$	$\mathbf{R}_Z(\gamma)=\mathbf{R}_Z(\delta)\mathbf{R}_Y(\beta)\mathbf{R}_X(\alpha)$
$\mathbf{R}_X(\gamma)=\mathbf{R}_Y(\beta)\mathbf{R}_X(\alpha)\mathbf{R}_Z(\delta)$	$\mathbf{R}_Y(\gamma)=\mathbf{R}_Y(\beta)\mathbf{R}_X(\alpha)\mathbf{R}_Z(\delta)$	$\mathbf{R}_Z(\gamma)=\mathbf{R}_Y(\beta)\mathbf{R}_X(\alpha)\mathbf{R}_Z(\delta)$

- Multiple Satisfying Values of (α, β, δ) for a Given γ Angle
- Tait-Bryan Increase the Rotation Operations that can be Reduced using Other Identities and Decompositions

Some Useful Identities for $SO(3)$ Elements

- Useful for Combining and Eliminating Elemental Rotations
- Choose Appropriate 3D Angles and Decomposition Types to Allow Advantageous use of Identities
- Use Other Identities (such as):
  - $\mathbf{R}_k(\alpha)\mathbf{R}_k(-\alpha) = \mathbf{I}$
  - $\mathbf{R}_\mathbf{X}(\alpha)\mathbf{R}_\mathbf{Z}(\pi) = \mathbf{R}_\mathbf{Z}(\pi)\mathbf{R}_\mathbf{X}(-\alpha)$
  - $\mathbf{R}_\mathbf{Y}(\alpha)\mathbf{R}_\mathbf{Z}(\pi) = \mathbf{R}_\mathbf{Z}(\pi)\mathbf{R}_\mathbf{Y}(-\alpha)$
  - $\mathbf{R}_k(\alpha)\mathbf{R}_k(\beta) = \mathbf{R}_k(\alpha+\beta)$
  - $\mathbf{R}_k(\pm 2m\pi) = \mathbf{R}_k(0) = \mathbf{I}$

$$k \in \{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$$

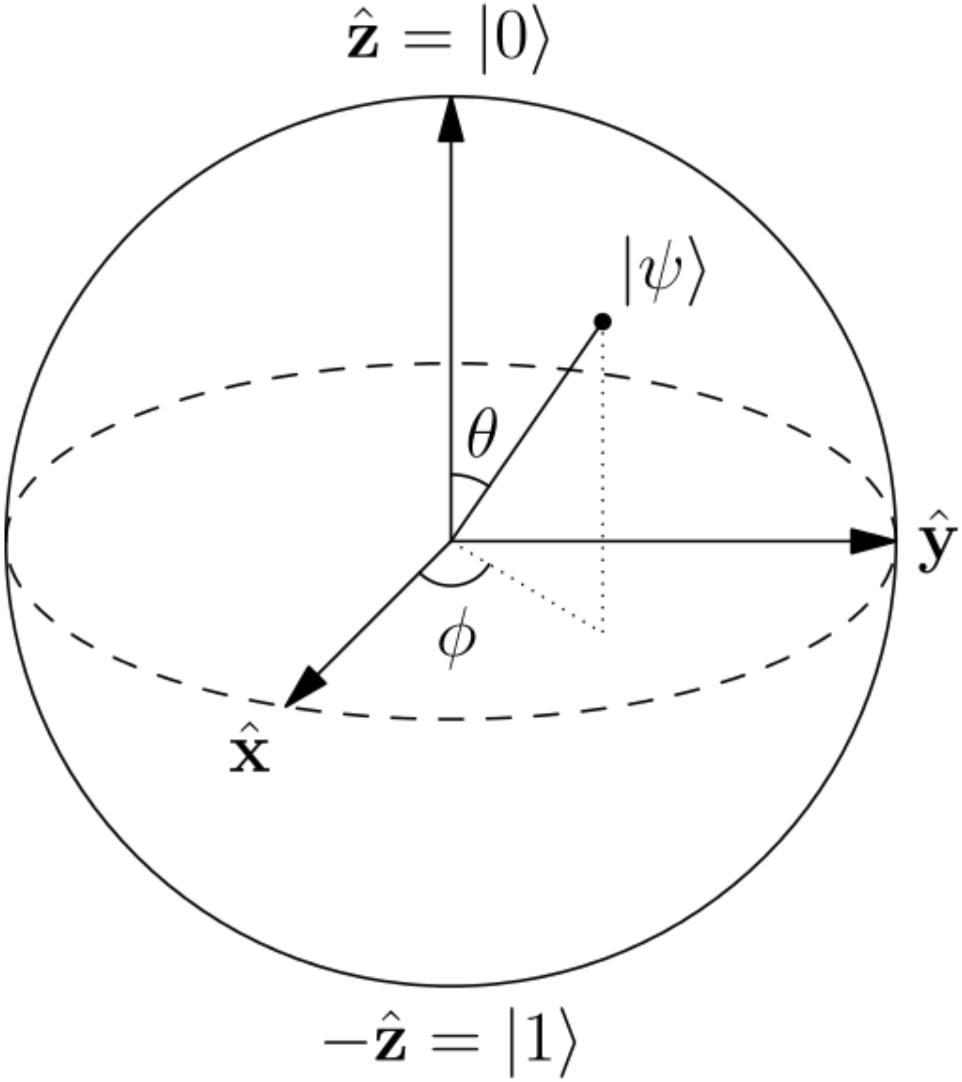


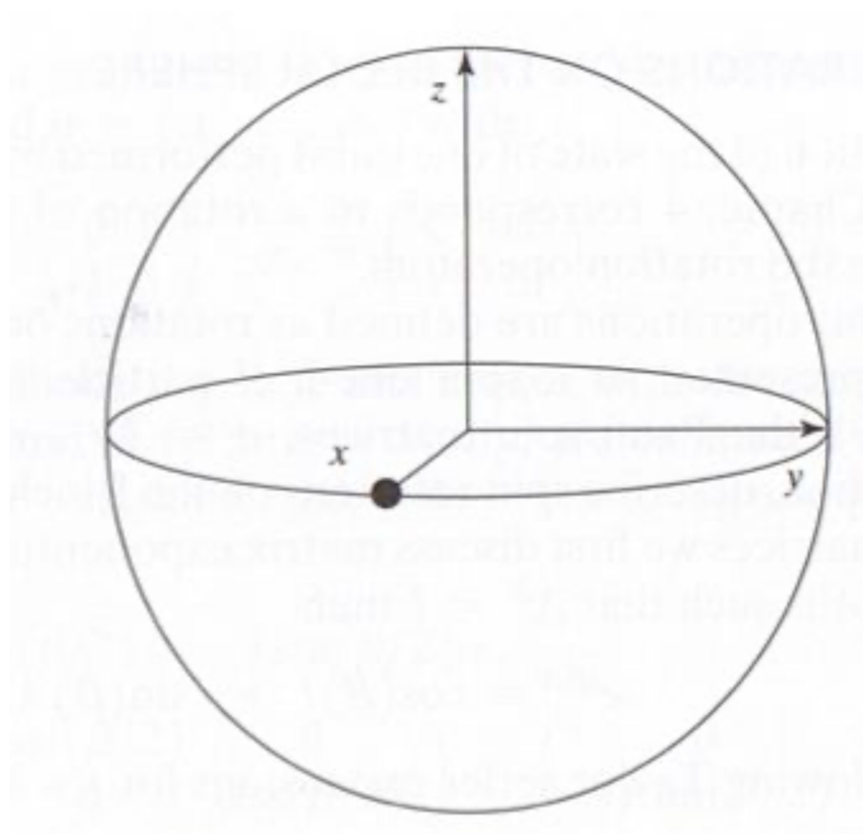
Quantum Logic Gates and Circuits

Bloch Sphere Geometry

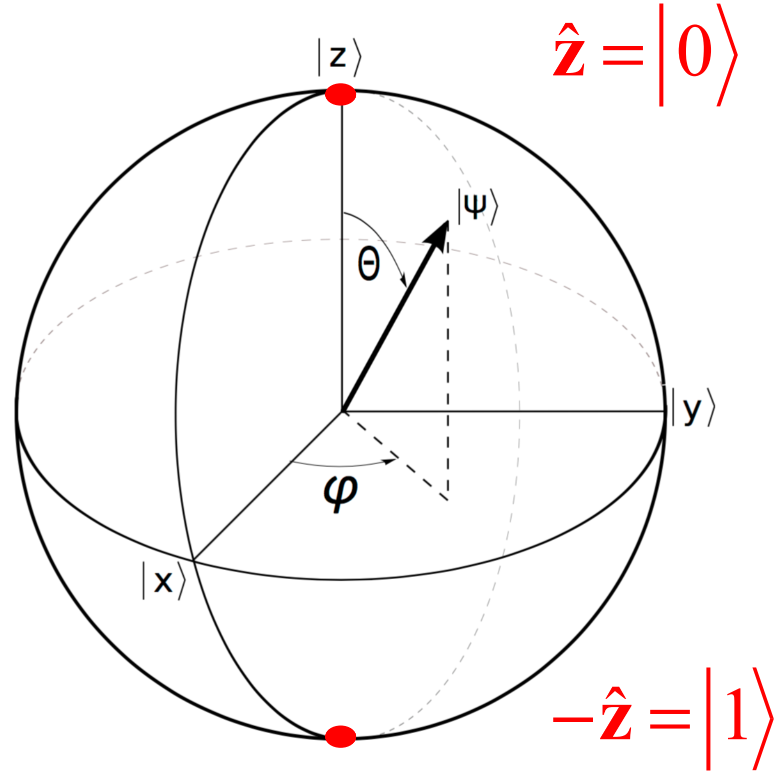
Bloch Sphere

- Geometric Representation of SINGLE qubit quantum state





Bloch Sphere

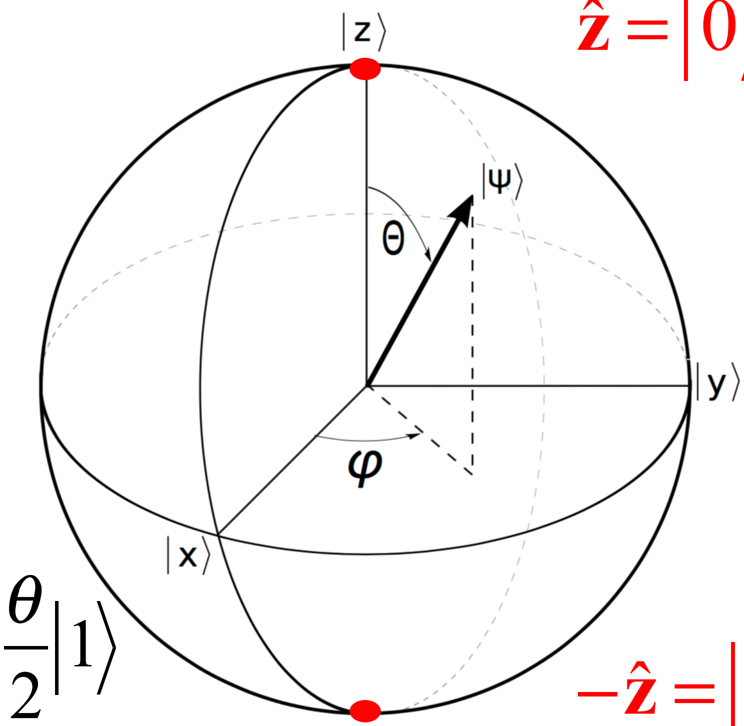


- Unit Radius Sphere
- Point on Surface Represents the State of a Qubit
- Qubit: Quantum State of Single "Information Carrier"
- The "Wave Function"
- Solution to Schrodinger Wave Equation Under Certain Assumptions
- Photonic Information Carriers include Location (spatial mode) and Polarization
- So-called "Computational Basis" Indicated in Red Font

Bloch Sphere

- Unit Radius Sphere
- Point on Surface Represents the State of a Qubit
- Qubit: Quantum State of Single "Information Carrier"
- The qubit:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$



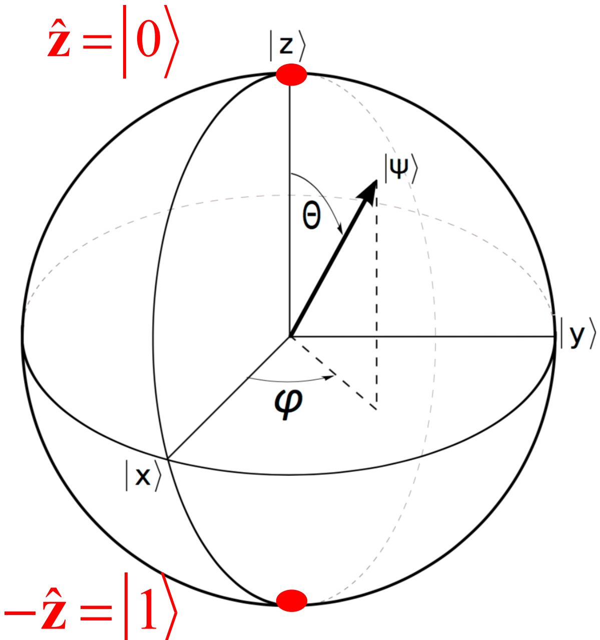
$$\hat{\mathbf{z}} = |0\rangle$$

$$-\hat{\mathbf{z}} = |1\rangle$$

- In General,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$$

Bloch Sphere as Qubit State Space



- Unit Radius Sphere
- ϕ: x-y Plane Angle Represents Phase
- θ: x-z Plane Angle Represents Superposition of Computation Basis ($\mathbf{Z}$-Basis)

- Point on Surface of Sphere Rectangular Coordinates:

$$\hat{\mathbf{n}} = n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}} + n_z\hat{\mathbf{z}} \qquad \hat{\mathbf{n}}\bullet\hat{\mathbf{x}} = \cos\varphi \qquad \hat{\mathbf{n}}\bullet\hat{\mathbf{y}} = \sin\theta \qquad \hat{\mathbf{n}}\bullet\hat{\mathbf{z}} = \cos\theta$$

$$\hat{\mathbf{n}} = \left(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\right)$$ Bloch Vector

$$= \hat{\mathbf{x}}\sin\theta\cos\varphi + \hat{\mathbf{y}}\sin\theta\sin\varphi + \hat{\mathbf{z}}\sin\theta\cos\varphi$$

$$= n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}} + n_z\hat{\mathbf{z}}$$

- Point on Surface in Spherical Coordinates: $\hat{\mathbf{n}} = \left(1, \theta, \varphi\right)$
- Phase ϕ “irrelevant” in Terms of Information Content

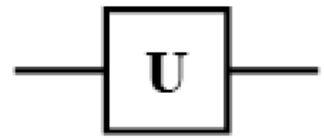
Quantum Logic Gates and Circuits

Physical Meaning of Single Qubit Gate

Moving on the Sphere Surface

Generic Single Qubit Gate

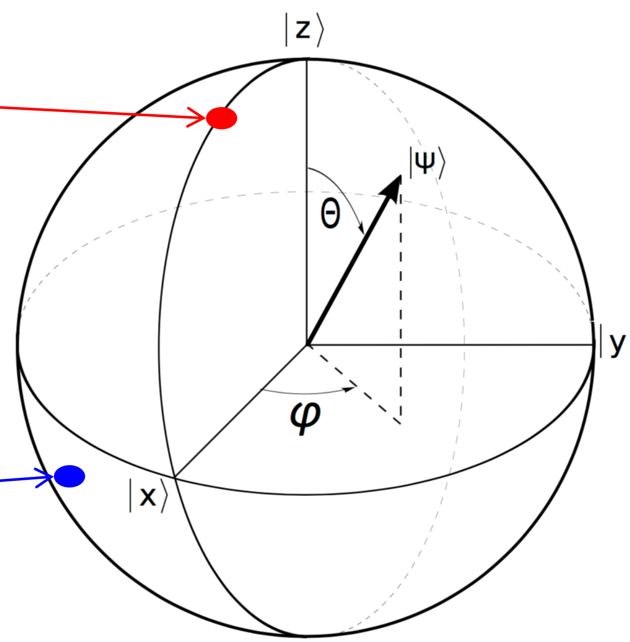
$$\mathbf{U} = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$$



- Denoted as $\mathbf{U}$ since Unitary Matrix
  - Hermitian: $\mathbf{U}^{\dagger} = \mathbf{U}$ ($\mathbf{U}^{\dagger}$ denotes the self-adjoint of $\mathbf{U}$)
    - Adjoint means transpose and conjugate
  - Self-Inverse: $\mathbf{U}^2 = \mathbf{I},\ \mathbf{U}^{-1} = \mathbf{U}$

$|\gamma_0\rangle$

"Ket Gamma-nought Evolves to Ket Gamma-sub-one"

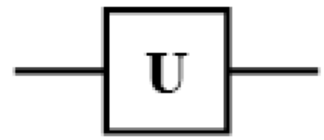


$$|\gamma_1\rangle = \mathbf{U}|\gamma_0\rangle$$

Schrodinger's Wave Equation

Generic Single Qubit Gate

$$\mathbf{U} = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$$



$$\mathcal{H}\left|\psi(\bar{r},t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\psi(\bar{r},t)\right\rangle$$

$$\mathcal{H}\left|\psi(t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle$$

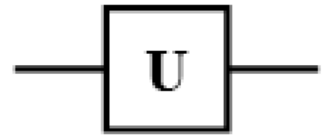
$$\left|\psi(t)\right\rangle = e^{-i\mathcal{H}t/\hbar}\left|\psi(0)\right\rangle$$

$$\mathbf{U} = e^{-i\mathcal{H}t/\hbar}$$

Relation of $\mathbf{U}$ to Qubit Wave Function

Generic Single Qubit Gate

$$\mathbf{U} = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$$



- Schrodinger's Time-varying Wave Equation,

$$\mathcal{H}\left|\psi(\bar{r},t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\psi(\bar{r},t)\right\rangle$$

$$\left|\psi(t)\right\rangle = e^{-i\mathcal{H}t/\hbar}\left|\psi(0)\right\rangle$$

$$\mathbf{U} = e^{-i\mathcal{H}t/\hbar}$$

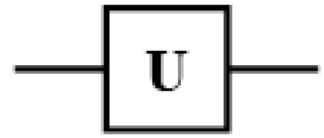
- Hamiltonian $\mathcal{H}$, is Self-adjoint
- $\mathbf{U}$ is Unitary since $\mathbf{U}^{\dagger}\mathbf{U}=\mathbf{I}$
- Hence, the Solution or Wave Function Evolves over Finite Time Interval (Gate Delay) as: $\left|\psi(t)\right\rangle = \mathbf{U}(t)\left|\psi(0)\right\rangle$
  - *i.e.,* a product of unitary operators is finite
- For Case of Time Independent Hamiltonian, $\mathcal{H}$: $\mathbf{U} = e^{-i\mathcal{H}t/\hbar}$

$$\left|\psi(t)\right\rangle = \mathbf{U}\left|\psi(0)\right\rangle \qquad \left|\psi_1\right\rangle = \mathbf{U}\left|\psi_0\right\rangle$$

Relation of **U** to Qubit Wave Function (cont.)

Generic Single Qubit Gate

$$\mathbf{U} = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$$



- Quantum Informatics:
  - Wave Function or Quantum State Represents Information
  - Quantum State of Single Carrier is a “Qubit”
  - Quantum State Evolution over Finite Time is a “Gate”
  - Gate is Represented by Unitary Operator, **U**
- Theoretical Model of Quantum Gate: $|\psi_1\rangle = \mathbf{U}|\psi_0\rangle$
- Example of General Single-gate Evolution:

$$|\psi_1\rangle = \mathbf{U}|\psi_0\rangle = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha u_{00} + u_{01}\beta \\ \alpha u_{10} + u_{11}\beta \end{bmatrix} = (\alpha u_{00} + u_{01}\beta)|0\rangle + (\alpha u_{10} + u_{11}\beta)|1\rangle$$

Quantum Informatics: Evolutions are Computations

- Series of Operators (Gates) over Time Evolve Qubits
- For Single Qubit, Conceptually is a Path on the Surface of Bloch Sphere
- Discretized as a Series of Locations at Discrete Times
  - Initial Location is Quantum State at Time Zero
  - Each Operation (Gate) “Moves” or “Rotates” Quantum State Vector to New Position on Bloch Sphere
  - Final Position is Result of Computation
- Series of Computations are Conceptually:
  - A Program on a Quantum Computer
  - A Cascade of Quantum Logic Gates
- Graphical Representation is a Quantum Logic Circuit or a Quantum Program

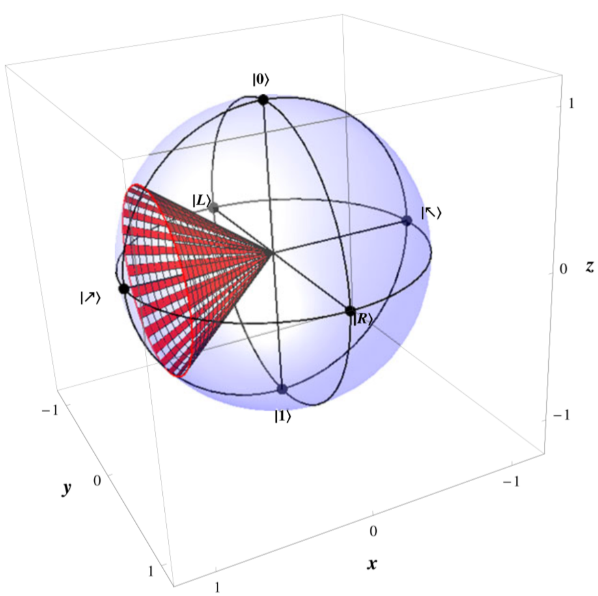
Moving on the Sphere Surface

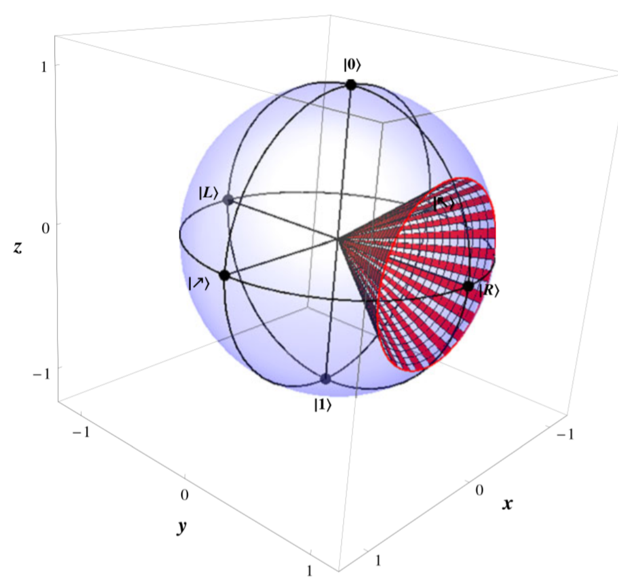
- Moving by Rotating About an Axis
- Defines a "Cone" on Surface About Each Axis:

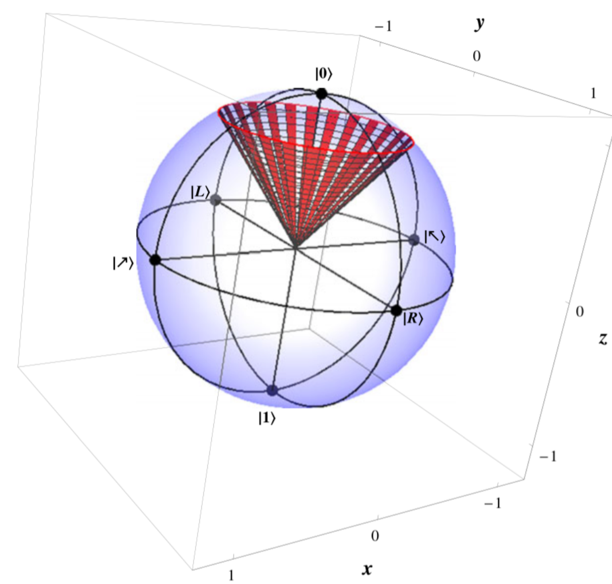
$$\mathbf{R_Y}(\alpha) = \begin{bmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{bmatrix}$$

$$\mathbf{R_X}(\alpha) = \begin{bmatrix} \cos(\alpha/2) & -i\sin(\alpha/2) \\ -i\sin(\alpha/2) & \cos(\alpha/2) \end{bmatrix}$$

$$\mathbf{R_Z}(\alpha) = \begin{bmatrix} e^{-i(\alpha/2)} & 0 \\ 0 & e^{i(\alpha/2)} \end{bmatrix}$$







Parameterized Bloch Sphere

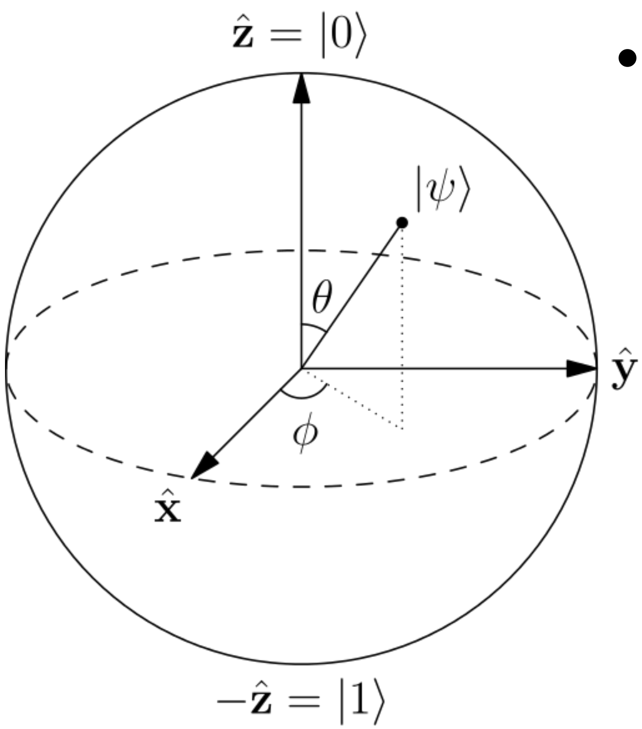
- Representing Qubits in terms of Bloch Sphere Angles

$$\left|\psi\right\rangle = \cos\left(\frac{\theta}{2}\right)\left|0\right\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)\left|1\right\rangle$$

$$= \cos\left(\frac{\theta}{2}\right)\left|0\right\rangle + \left(\cos\phi + i\sin\phi\right)\sin\left(\frac{\theta}{2}\right)\left|1\right\rangle$$

$$0 \le \theta \le \pi$$

$$0 \le \phi \le 2\pi$$



- Qubits can have an Arbitrary Phase Shift, γ, that is Irrelevant and is NOT Represented on Bloch Sphere

$$\left|\psi\right\rangle = e^{i\gamma}\left(\cos\left(\frac{\theta}{2}\right)\left|0\right\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)\left|1\right\rangle\right)$$

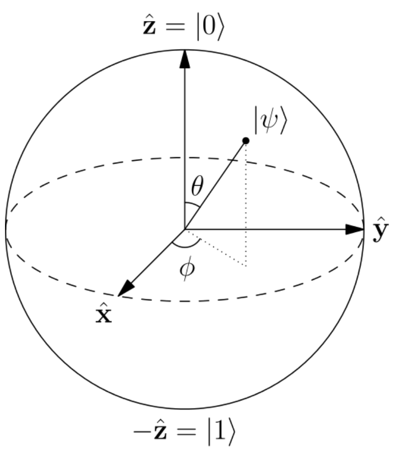
$$\Rightarrow \cos\left(\frac{\theta}{2}\right)\left|0\right\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)\left|1\right\rangle$$

Parameterized Bloch Sphere

- Unitary Transfer Matrix that Causes Qubit to Rotate to New Position on Bloch Sphere through a Time-based Evolution can be Parameterized by Bloch Sphere Angles

$$0 \le \theta \le \pi$$

$$0 \le \phi \le 2\pi$$



$$\mathbf{R}(\theta,\phi) = |\psi\rangle\langle\psi| = \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix} \otimes \begin{bmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+\cos\theta & \cos\phi\sin\theta - i\sin\phi\sin\theta \\ \cos\phi\sin\theta + i\sin\phi\cos\theta & 1-\cos\theta \end{bmatrix}$$

Quantum Logic Gates and Circuits

The Pauli Gates

Parameterized Bloch Sphere

- General Rotation Matrix $\mathbf{R}(\theta,\phi)$ can be Expressed in Terms of the Pauli Matrices as:

$$\mathbf{R}(\theta,\phi)=|\psi\rangle\langle\psi|=\frac{1}{2}\left(\mathbf{I}+\mathbf{X}\cos\phi\sin\theta+\mathbf{Y}\sin\phi\sin\theta+\mathbf{Z}\cos\theta\right)=\frac{1}{2}\left(\mathbf{I}+\vec{\mathbf{n}}\bullet\vec{\boldsymbol{\sigma}}\right)$$

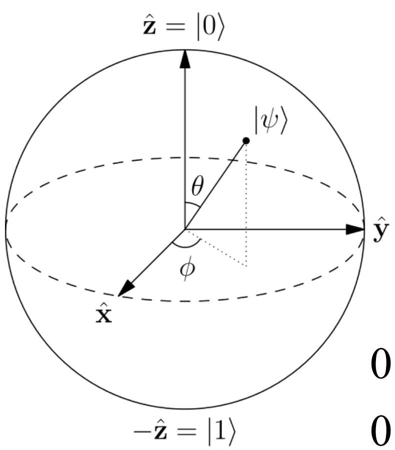
- Position of Qubit is (unit) Bloch Vector

$$\vec{\mathbf{n}}=\hat{\mathbf{n}}=\left(n_x,n_y,n_z\right)=\left(\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta\right)=n_x\hat{\mathbf{x}}+n_y\hat{\mathbf{y}}+n_z\hat{\mathbf{z}}$$

- The "Pauli Vector" is a Vector of Matrices:

$$\vec{\boldsymbol{\sigma}}=\hat{\boldsymbol{\sigma}}=\left(\mathbf{X},\mathbf{Y},\mathbf{Z}\right)$$

- The Pauli Matrices:



$$\mathbf{X}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\quad\mathbf{Y}=\begin{bmatrix}0&-i\\i&0\end{bmatrix}\quad\mathbf{Z}=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$0\le\theta\le\pi$$

$$0\le\phi\le2\pi$$

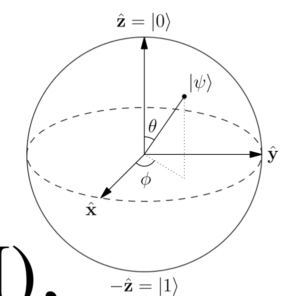
Parameterized Bloch Sphere

- The Bloch Vector is Rotated about the Axes in the Bloch Sphere via the Following Rotation Matrices:

$$\mathbf{R}_x(\theta) = e^{-i\frac{\theta}{2}\mathbf{X}} \qquad \mathbf{R}_y(\theta) = e^{-i\frac{\theta}{2}\mathbf{Y}} \qquad \mathbf{R}_z(\theta) = e^{-i\frac{\theta}{2}\mathbf{Z}}$$

- Since the Pauli Matrices are Unitary (*eg*. $\mathbf{A}^2 = \mathbf{I}$), can use Euler's Equation in Matrix Form:

$$e^{\pm i\theta\mathbf{A}} = \cos(\theta)\mathbf{I} \pm i\sin(\theta)\mathbf{A}$$



$$0 \le \theta \le \pi$$
$$0 \le \phi \le 2\pi$$

- Since the Pauli Matrices Satisfy $\mathbf{X}^2 = \mathbf{Y}^2 = \mathbf{Z}^2 = \mathbf{I}$), Rotation Operators Become:

$$\mathbf{R}_x(\theta) = e^{-i\frac{\theta}{2}\mathbf{X}} = \cos\frac{\theta}{2}\mathbf{I} - i\sin\frac{\theta}{2}\mathbf{X} = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

Parameterized Bloch Sphere

- Since the Pauli Matrices Satisfy $\mathbf{X}^2=\mathbf{Y}^2=\mathbf{Z}^2=\mathbf{I}$, Rotation Operators Become:

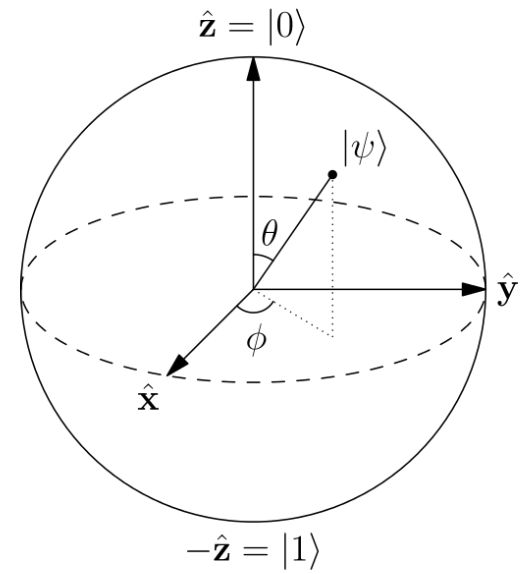
$$\mathbf{R}_x(\theta)=e^{-i\frac{\theta}{2}\mathbf{X}}=\cos\frac{\theta}{2}\mathbf{I}-i\sin\frac{\theta}{2}\mathbf{X}=\begin{bmatrix}\cos\frac{\theta}{2} & -i\sin\frac{\theta}{2}\\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{bmatrix}$$

$$\mathbf{R}_y(\theta)=e^{-i\frac{\theta}{2}\mathbf{Y}}=\cos\frac{\theta}{2}\mathbf{I}-i\sin\frac{\theta}{2}\mathbf{Y}=\begin{bmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{bmatrix}$$

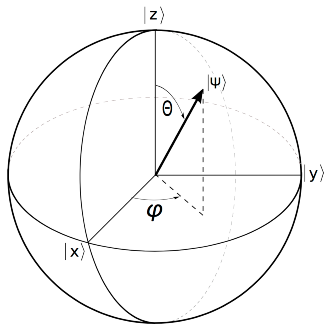
$$\mathbf{R}_z(\theta)=e^{-i\frac{\theta}{2}\mathbf{Z}}=\cos\frac{\theta}{2}\mathbf{I}-i\sin\frac{\theta}{2}\mathbf{Z}=\begin{bmatrix}e^{-i\frac{\theta}{2}} & 0\\ 0 & e^{i\frac{\theta}{2}}\end{bmatrix}$$

$$0\le\theta\le\pi$$

$$0\le\phi\le 2\pi$$



The Pauli Gates: Rotation Angle is π



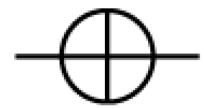
- The Trivial Case:

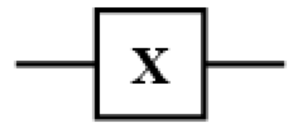
$$\sigma_0 = \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{array}{l} |0\rangle \rightarrow |0\rangle \\ |1\rangle \rightarrow |1\rangle \end{array}$$

- Pauli-$\mathbf{X}$ Rotates About $\mathbf{X}$ by π Radians:

$$\sigma_1 = \sigma_{\mathbf{X}} = \mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{array}{l} |0\rangle \rightarrow |1\rangle \\ |1\rangle \rightarrow |0\rangle \end{array}$$

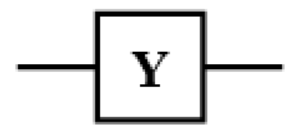
"NOT"
"bit flip"





- Pauli-$\mathbf{Y}$ Rotates About $\mathbf{Y}$ by π Radians:

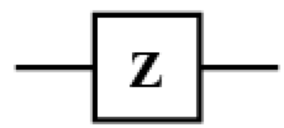
$$\sigma_2 = \sigma_{\mathbf{Y}} = \mathbf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \begin{array}{l} |0\rangle \rightarrow i|1\rangle \\ |1\rangle \rightarrow -i|0\rangle \end{array}$$



- Pauli-$\mathbf{Z}$ Rotates About $\mathbf{Z}$ by π Radians:

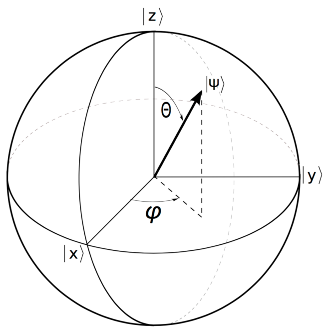
$$\sigma_3 = \sigma_{\mathbf{z}} = \mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{array}{l} |0\rangle \rightarrow |0\rangle \\ |1\rangle \rightarrow -|1\rangle \end{array}$$

"phase flip"



$$\mathbf{X}^2 = \mathbf{Y}^2 = \mathbf{Z}^2 = -i\mathbf{XYZ} = \mathbf{I}$$

Parameterized Rotation



- Arbitrary Rotation About Axis j:

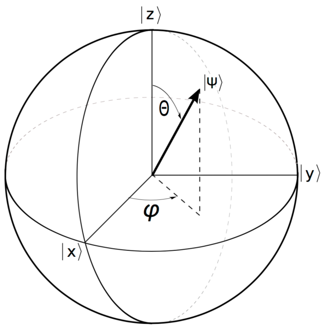
$$\mathbf{J}_{\alpha} = \cos\alpha\mathbf{I} + i\sin\alpha\mathbf{J}$$

- j Axis is $x, y,$ or z Axis
- $\mathbf{J} = \mathbf{X}, \mathbf{Y}, \text{or } \mathbf{Z}$
- Theoretical “Gate” or Operator
- Bloch Sphere is Unit Radius
- Point on Surface of Sphere Rectangular Coordinates:

$$\hat{\mathbf{n}} = n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}} + n_z\hat{\mathbf{z}}$$

$$\hat{\mathbf{n}} \bullet \hat{\mathbf{x}} = \cos\varphi \qquad \hat{\mathbf{n}} \bullet \hat{\mathbf{y}} = \sin\theta \qquad \hat{\mathbf{n}} \bullet \hat{\mathbf{z}} = \cos\theta$$

Moving on the Sphere Surface



Using Pauli Gates for General Rotations

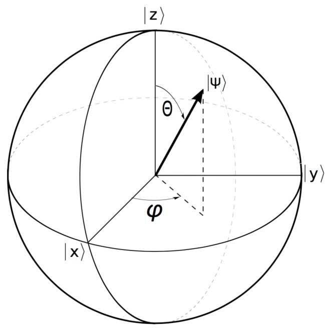
- We can Move Anywhere by using Pauli Rotations
  - And one more operator

$$\widehat{\mathbf{n}} = n_x \hat{x} + n_y \hat{y} + n_z \hat{z}$$ *Bloch Vector*

$$\vec{\boldsymbol{\sigma}} = \left(\sigma_x, \sigma_y, \sigma_z\right) = \mathbf{X}\hat{\mathbf{x}} + \mathbf{Y}\hat{\mathbf{y}} + \mathbf{Z}\hat{\mathbf{z}}$$ *Pauli Vector*

$$S(\widehat{\mathbf{n}}) = \widehat{\mathbf{n}} \bullet \vec{\boldsymbol{\sigma}} = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z$$

The Pauli Gate Algebra



$$\mathbf{XY} = -\mathbf{YZ} = i\mathbf{Z}$$

$$\mathbf{YZ} = -\mathbf{ZY} = i\mathbf{X}$$

$$\mathbf{ZX} = -\mathbf{XZ} = i\mathbf{Y}$$

$$\mathbf{H} = \frac{\mathbf{X}+\mathbf{Z}}{\sqrt{2}}$$

$$\mathbf{HXH} = \mathbf{Z}$$

$$\mathbf{HYH} = -\mathbf{Y}$$

$$\mathbf{HZH} = \mathbf{X}$$

$$\mathbf{X}^2 = \mathbf{Y}^2 = \mathbf{Z}^2 = \mathbf{H}^2 = \mathbf{I}$$

$$\mathbf{XXX} = \mathbf{X}$$

$$\mathbf{XYX} = -\mathbf{Y}$$

$$\mathbf{XZX} = -\mathbf{Z}$$

$$\mathbf{X}^2 = \mathbf{Y}^2 = \mathbf{Z}^2 = -i\mathbf{XYZ} = \mathbf{I}$$

$$\mathbf{I} = -i\mathbf{XYZ}$$

$$(\mathbf{I})\mathbf{Z} = (-i\mathbf{XYZ})\mathbf{Z}$$

$$\mathbf{Z} = -i\mathbf{XY}$$

$$(\mathbf{Z})\mathbf{Y} = (-i\mathbf{XY})\mathbf{Y}$$

$$\mathbf{ZY} = -i\mathbf{X}$$

$$(\mathbf{ZY})\mathbf{X} = (-i\mathbf{X})\mathbf{X}$$

$$\mathbf{ZYX} = -i\mathbf{I}$$

$$\mathbf{Z}(\mathbf{ZYX}) = \mathbf{Z}(-i\mathbf{I})$$

$$\mathbf{YX} = -i\mathbf{Z}$$

Pauli Operators (Theory)

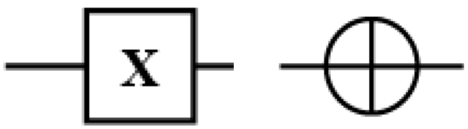
- Pauli Group $\mathbb{P}$ is Set of Pauli Operators with Coefficients $\{\pm 1, \pm i\}$
- Single Qubit Pauli Group $\mathbb{P}_{\mathbf{1}}$ is:

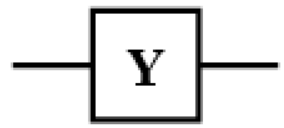
$$\mathbb{P}_1 = \left\{ \mathbf{I}, \pm\mathbf{X}, \pm\mathbf{Y}, \pm\mathbf{Z}, \pm i\mathbf{I}, \pm i\mathbf{X}, \pm i\mathbf{Y}, \pm i\mathbf{Z} \right\} = \left\{ \mathbf{P} \in \mathbb{P}_1 \right\}$$

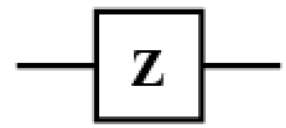
- Multi-qubit (n) Pauli Group Consists of Elements that are Products of n Pauli Operators
- Clifford Group, $\mathbf{C}=\{\mathbf{C}_i \in \mathbf{C}\}$, is Group of Transformations that Leave the Pauli Group Invariant $\mathbf{CPC}^\dagger = \mathbf{P}'$ where $\mathbf{P}' \in \mathbb{P}_1$ with
- Prominent Members of Clifford Group are: Hadamard ($\mathbf{H}$), Phase ($\mathbf{S}$), and Controlled-NOT (**CNOT**, Feynman, Controlled-$\mathbf{X}$)
- Gates in $\mathbf{P}$ and $\mathbf{C}$ are not Universal Unless $\mathbf{T}$ Included

Euler Elemental Rotation Decomposition Examples for Pauli Operators

- Global Phase Shifts of δ are Represented as $(e^{i\delta})\mathbf{I}$







$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathbf{X} = \mathbf{R_X}(\pi)\left(e^{i\frac{\pi}{2}}\right)\mathbf{I} \qquad \mathbf{Y} = \mathbf{R_Y}(\pi)\left(e^{i\frac{\pi}{2}}\right)\mathbf{I} \qquad \mathbf{Z} = \mathbf{R_Z}(\pi)\left(e^{i\frac{\pi}{2}}\right)\mathbf{I}$$

$$\mathbf{X} = \mathbf{R_Z}(\pi)\mathbf{R_Y}(\pi)\left(e^{-i\frac{\pi}{2}}\right)\mathbf{I} \qquad \mathbf{Y} = \mathbf{R_Z}(\pi)\mathbf{R_X}(\pi)\left(e^{-i\frac{\pi}{2}}\right)\mathbf{I} \qquad \mathbf{Z} = \mathbf{R_X}(\pi)\mathbf{R_Y}(\pi)\left(e^{i\frac{\pi}{2}}\right)\mathbf{I}$$

$$\mathbf{X} = \mathbf{R_Y}(\pi)\mathbf{R_Z}(\pi)\left(e^{i\frac{\pi}{2}}\right)\mathbf{I} \qquad \mathbf{Y} = \mathbf{R_X}(\pi)\mathbf{R_Z}(\pi)\left(e^{i\frac{\pi}{2}}\right)\mathbf{I} \qquad \mathbf{Z} = \mathbf{R_Y}(\pi)\mathbf{R_X}(\pi)\left(e^{-i\frac{\pi}{2}}\right)\mathbf{I}$$

$$\mathbf{X} = \mathbf{R_Z}\left(-\frac{\pi}{2}\right)\mathbf{R_Y}(\pi)\mathbf{R_Z}\left(\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{2}}\right)\mathbf{I} \qquad \mathbf{Y} = \mathbf{R_Z}\left(\frac{\pi}{2}\right)\mathbf{R_X}(\pi)\mathbf{R_Z}\left(-\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{2}}\right)\mathbf{I} \qquad \mathbf{Z} = \mathbf{R_X}\left(\frac{\pi}{2}\right)\mathbf{R_Y}(\pi)\mathbf{R_X}\left(-\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{2}}\right)\mathbf{I}$$

$$\mathbf{X} = \mathbf{R_Y}\left(\frac{\pi}{2}\right)\mathbf{R_Z}(\pi)\mathbf{R_Y}\left(-\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{2}}\right)\mathbf{I} \qquad \mathbf{Y} = \mathbf{R_X}\left(-\frac{\pi}{2}\right)\mathbf{R_Z}(\pi)\mathbf{R_X}\left(\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{2}}\right)\mathbf{I} \qquad \mathbf{Z} = \mathbf{R_Y}\left(-\frac{\pi}{2}\right)\mathbf{R_X}(\pi)\mathbf{R_Y}\left(\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{2}}\right)\mathbf{I}$$

Quantum Logic Gates and Circuits

Single Qubit Gates that Modify the Probability Amplitude or Produce Pure (Relative) Phase Shifts

Hadamard Operator

- This Operator is Commonly used to Maximize Superposition of a Qubit in a Basis State

$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Example: $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$

$$\mathbf{H}|\psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \frac{\alpha_0}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{\alpha_1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Hadamard Operator

- This Operator is Commonly used to Maximize Superposition of a Qubit in a Basis State

$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Example: $|\psi\rangle = 0|0\rangle + 1|1\rangle = |1\rangle$

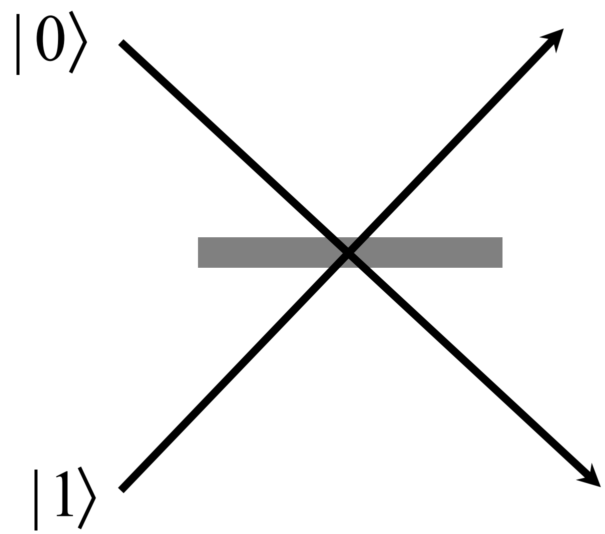
$$\mathbf{H}|\psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = (1/\sqrt{2})|0\rangle - (1/\sqrt{2})|1\rangle$$

$$\text{Prob}[|0\rangle \text{ measured}] = (1/\sqrt{2})^2 = 50\%$$

$$\text{Prob}[|1\rangle \text{ measured}] = (1/\sqrt{2})^2 = 50\%$$

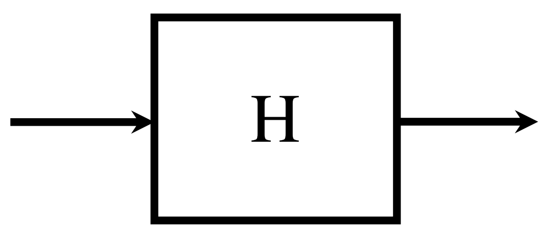
Beam Splitter Example

- 50-50 Beam Splitter Performs a Hadamard Transform on Particles (location/spatially encoded information)
- Beam Splitters have been Constructed for Quantum Particles other than Photons

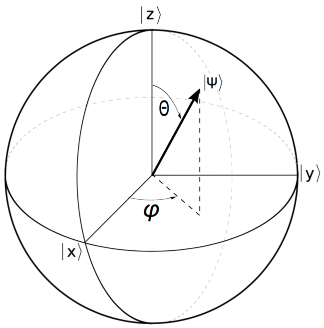


$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

$$|\varphi\rangle = \mathbf{H} |\psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \alpha_0 + \alpha_1 \\ \alpha_0 - \alpha_1 \end{bmatrix}$$

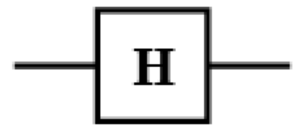


Other Single-Qubit Gates



- Hadamard (90-degree rotation about axis parallel to the *xy*-plane)

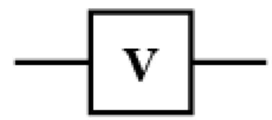
$$\mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad |0\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad |1\rangle \rightarrow \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



$$\mathbf{H} = \mathbf{R_X}(\pi)\mathbf{R_Y}\left(\frac{\pi}{2}\right)\mathbf{Ph}\left(\frac{\pi}{2}\right) \qquad \mathbf{H} = \mathbf{R_Y}\left(\frac{\pi}{2}\right)\mathbf{R_Z}(\pi)\mathbf{Ph}\left(\frac{\pi}{2}\right)$$

- Square Root of NOT (Square root of $\mathbf{X}$):

$$\mathbf{V} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \quad |0\rangle \rightarrow \frac{(1+i)|0\rangle + (1-i)|1\rangle}{2} \quad |1\rangle \rightarrow \frac{(1-i)|0\rangle + (1+i)|1\rangle}{2}$$

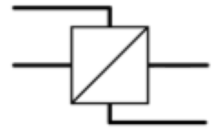


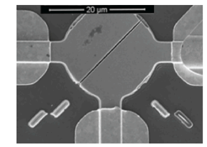
$$\mathbf{V} = \mathbf{R_X}\left(\frac{\pi}{2}\right)\mathbf{Ph}\left(\frac{\pi}{4}\right) \qquad \mathbf{V} = \mathbf{R_Z}\left(-\frac{\pi}{2}\right)\mathbf{R_Y}\left(\frac{\pi}{2}\right)\mathbf{R_Z}\left(-\frac{\pi}{2}\right)\mathbf{Ph}\left(\frac{\pi}{4}\right)$$

Euler Elemental Rotation Decomposition Examples for Hadamard Operator

- Global Phase Shifts of δ are Represented as $(e^{i\delta}\mathbf{I})$

$$\mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$





SMU Photonic Quantum Circuit

(x-z-x)

$$\mathbf{H} = \mathbf{R}_X\left(\frac{\pi}{2}\right)\mathbf{R}_Z\left(\frac{\pi}{2}\right)\mathbf{R}_X\left(\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{2}}\mathbf{I}\right)$$

$$\mathbf{H} = \mathbf{R}_X\left(-\frac{\pi}{2}\right)\mathbf{R}_Z\left(-\frac{\pi}{2}\right)\mathbf{R}_X\left(-\frac{\pi}{2}\right)\left(e^{-i\frac{\pi}{2}}\mathbf{I}\right)$$

(y-x-y)

$$\mathbf{H} = \mathbf{R}_X(\pi)\mathbf{R}_Y\left(\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{2}}\mathbf{I}\right)$$

$$\mathbf{H} = \mathbf{R}_Y(\pi)\mathbf{R}_X(\pi)\mathbf{R}_Y\left(-\frac{\pi}{2}\right)\left(e^{-i\frac{\pi}{2}}\mathbf{I}\right)$$

(y-z-y)

$$\mathbf{H} = \mathbf{R}_Y(\pi)\mathbf{R}_Z(\pi)\mathbf{R}_Y\left(\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{2}}\mathbf{I}\right)$$

$$\mathbf{H} = \mathbf{R}_Z(\pi)\mathbf{R}_Y\left(-\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{2}}\mathbf{I}\right)$$

$$\mathbf{H} = \mathbf{R}_Y\left(\frac{\pi}{2}\right)\mathbf{R}_Z(\pi)\left(e^{i\frac{\pi}{2}}\mathbf{I}\right)$$

(z-x-z)

$$\mathbf{H} = \mathbf{R}_Z\left(\frac{\pi}{2}\right)\mathbf{R}_X\left(\frac{\pi}{2}\right)\mathbf{R}_Z\left(\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{2}}\mathbf{I}\right)$$

$$\mathbf{H} = \mathbf{R}_Z\left(-\frac{\pi}{2}\right)\mathbf{R}_X\left(-\frac{\pi}{2}\right)\mathbf{R}_Z\left(-\frac{\pi}{2}\right)\left(e^{-i\frac{\pi}{2}}\mathbf{I}\right)$$

Single Qubit Operations (Square Root of X)

Transformation Matrix:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\pi/4} & e^{-i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4} \end{bmatrix}$$

From Euler's Identity:

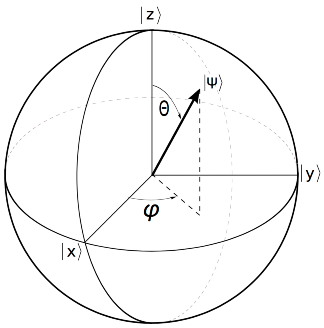
$$e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i) \qquad e^{-i\pi/4} = \frac{1}{\sqrt{2}}(1-i)$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\pi/4} & e^{-i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

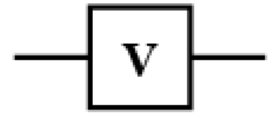
Two Gates in Series (Square of Matrix):

$$\left(\frac{1}{2}\right)^2 \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}^2 = \frac{1}{4}\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Other Single-Qubit Gates

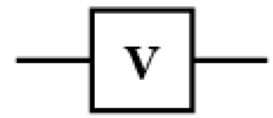


$$\mathbf{V} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$



- Square Root of NOT (Square root of X):

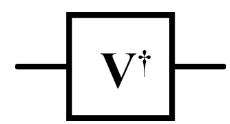
$$\mathbf{V} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \quad |0\rangle \rightarrow \frac{(1+i)|0\rangle + (1-i)|1\rangle}{2} \quad |1\rangle \rightarrow \frac{(1-i)|0\rangle + (1+i)|1\rangle}{2}$$



$$\mathbf{V} = \mathbf{R_X}\left(\frac{\pi}{2}\right)\mathbf{Ph}\left(\frac{\pi}{4}\right) \qquad \mathbf{V} = \mathbf{R_Z}\left(-\frac{\pi}{2}\right)\mathbf{R_Y}\left(\frac{\pi}{2}\right)\mathbf{R_Z}\left(-\frac{\pi}{2}\right)\mathbf{Ph}\left(\frac{\pi}{4}\right)$$

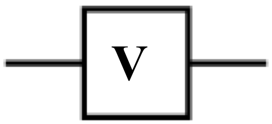
- Another Square Root of NOT Gate:

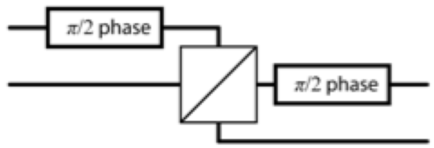
$$\mathbf{V}^{\dagger} = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \quad |0\rangle \rightarrow \frac{(1-i)|0\rangle + (1+i)|1\rangle}{2} \quad |1\rangle \rightarrow \frac{(1+i)|0\rangle + (1-i)|1\rangle}{2}$$

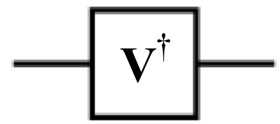


Euler Elemental Rotation Decomposition Examples for Square-root of NOT Gates

- Global Phase Shifts of δ are Represented as $(e^{i\delta})\mathbf{I}$







$$\mathbf{V} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

$$\mathbf{V}^{\dagger} = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

$$\mathbf{V} = \mathbf{R}_{\mathbf{X}}\left(\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{4}}\mathbf{I}\right)$$

$$\mathbf{V}^{\dagger} = \mathbf{R}_{\mathbf{X}}\left(-\frac{\pi}{2}\right)\left(e^{-i\frac{\pi}{4}}\mathbf{I}\right)$$

$$\mathbf{V} = \mathbf{R}_{\mathbf{Z}}\left(-\frac{\pi}{2}\right)\mathbf{R}_{\mathbf{Y}}\left(\frac{\pi}{2}\right)\mathbf{R}_{\mathbf{Z}}\left(\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{4}}\mathbf{I}\right)$$

$$\mathbf{V}^{\dagger} = \mathbf{R}_{\mathbf{Z}}\left(-\frac{\pi}{2}\right)\mathbf{R}_{\mathbf{Y}}\left(-\frac{\pi}{2}\right)\mathbf{R}_{\mathbf{Z}}\left(\frac{\pi}{2}\right)\left(e^{-i\frac{\pi}{4}}\mathbf{I}\right)$$

$$\mathbf{V} = \mathbf{R}_{\mathbf{Y}}\left(\frac{\pi}{2}\right)\mathbf{R}_{\mathbf{Z}}\left(\frac{\pi}{2}\right)\mathbf{R}_{\mathbf{Y}}\left(-\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{4}}\mathbf{I}\right)$$

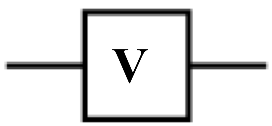
$$\mathbf{V}^{\dagger} = \mathbf{R}_{\mathbf{Y}}\left(\frac{\pi}{2}\right)\mathbf{R}_{\mathbf{Z}}\left(-\frac{\pi}{2}\right)\mathbf{R}_{\mathbf{Y}}\left(-\frac{\pi}{2}\right)\left(e^{-i\frac{\pi}{4}}\mathbf{I}\right)$$

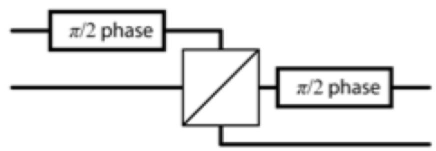
$$\mathbf{V} = \mathbf{R}_{\mathbf{Z}}\left(-\frac{\pi}{2}\right)\mathbf{H}\mathbf{R}_{\mathbf{Z}}\left(-\frac{\pi}{2}\right)\left(e^{-i\frac{\pi}{4}}\mathbf{I}\right)$$

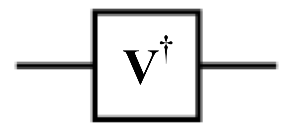
(Example of **V** in Terms of **H** Operator)

Decomposition Examples for Square-root of NOT Gates with Hadamard

- Global Phase Shifts of δ are Represented as $(e^{i\delta})\mathbf{I}$







$$\mathbf{V} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \qquad \mathbf{V}^{\dagger} = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

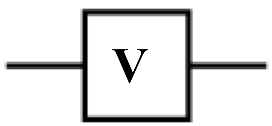
$$\mathbf{V} = \mathbf{R_z}\left(-\frac{\pi}{2}\right)\mathbf{H}\mathbf{R_z}\left(-\frac{\pi}{2}\right)\left(e^{-i\frac{\pi}{4}}\mathbf{I}\right)$$

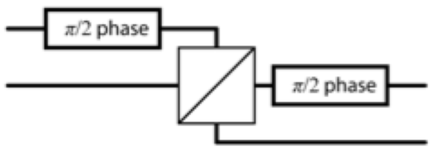
$$\mathbf{V} = \mathbf{HSH} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

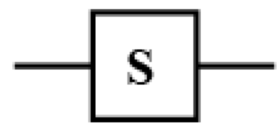
$$\mathbf{V}^{\dagger} = \mathbf{HS}^{\dagger}\mathbf{H} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

Decomposition Examples for Square-root of NOT Gates with S

- Global Phase Shifts of δ are Represented as $(e^{i\delta})\mathbf{I}$







$$\mathbf{V} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \qquad \mathbf{H} = \left(e^{-i\frac{\pi}{4}}\right)\mathbf{SVS} \qquad \mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

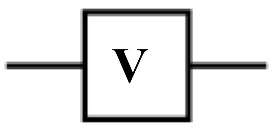
$$\mathbf{SVS} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1+i & i(1-i) \\ 1-i & i(1+i) \end{bmatrix}$$

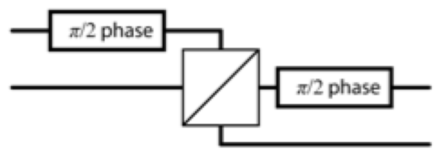
$$= \frac{1}{2}\begin{bmatrix} 1+i & i(1-i) \\ i(1-i) & -(1+i) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i(1-i) & i(1-i) \\ i(1-i) & -i(1-i) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ 1+i & -(1+i) \end{bmatrix}$$

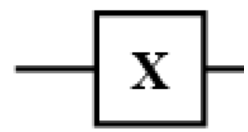
$$= \left(\frac{1}{2} + i\frac{1}{2}\right)\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \left(e^{-i\frac{\pi}{4}}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \left(e^{-i\frac{\pi}{4}}\right)\mathbf{H}$$

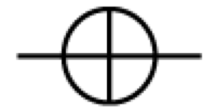
Decomposition Examples for Square-root of NOT Gates with X

- Global Phase Shifts of δ are Represented as $(e^{i\delta})\mathbf{I}$









$$\mathbf{V} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{VX} = \mathbf{V}^{\dagger}$$

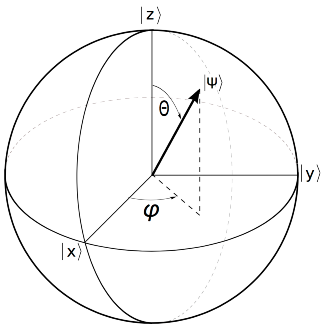
$$\mathbf{V}^{\dagger}\mathbf{X} = \mathbf{V}$$

$$\mathbf{VX} = \left(\frac{1}{2}\right)\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \left(\frac{1}{2}\right)\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} = \mathbf{V}^{\dagger}$$

$$\mathbf{V}^{\dagger}\mathbf{X} = \left(\frac{1}{2}\right)\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \left(\frac{1}{2}\right)\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} = \mathbf{V}$$

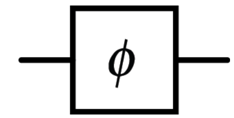
Single-Qubit Phase Shift Gates

Fixed Rotations About z-axis



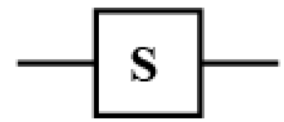
- General Case (no effect on state value):

$$\mathbf{P}_\phi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix} \qquad |0\rangle \rightarrow |0\rangle \qquad |1\rangle \rightarrow e^{i\phi}|1\rangle$$



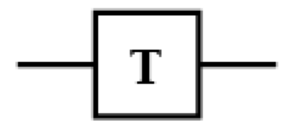
- Phase Shift by π/2 (Phase Gate):

$$\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = e^{i(\pi/4)} \begin{bmatrix} e^{-i(\pi/4)} & 0 \\ 0 & e^{i(\pi/4)} \end{bmatrix} \qquad \mathbf{S}^2 = \mathbf{Z}$$



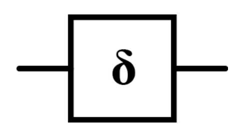
- Phase Shift by π/4 ("π/8 Gate" or "22.5° gate"):

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i(\pi/4)} \end{bmatrix} = e^{i(\pi/8)} \begin{bmatrix} e^{-i(\pi/8)} & 0 \\ 0 & e^{i(\pi/8)} \end{bmatrix} \qquad \mathbf{T}^2 = \mathbf{S}$$



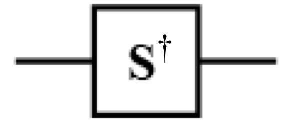
- General Phase Shift by δ:

$$\mathbf{Ph}_\delta = e^{i\delta} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad |0\rangle \rightarrow e^{i\delta}|0\rangle \qquad |1\rangle \rightarrow e^{i\delta}|1\rangle$$



Euler Elemental Rotation Decomposition Examples for Relative Phase Shift Gates

- Global Phase Shifts of δ are Represented as $(e^{i\delta}\mathbf{I})$



$$\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad \mathbf{S}^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

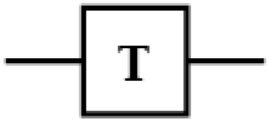
$$\mathbf{S} = \mathbf{R_Z}\left(\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{4}}\mathbf{I}\right) \qquad \mathbf{S}^\dagger = \mathbf{R_Z}\left(-\frac{\pi}{2}\right)\left(e^{-i\frac{\pi}{4}}\mathbf{I}\right)$$

$$\mathbf{S} = \mathbf{R_X}\left(\frac{\pi}{2}\right)\mathbf{R_Y}\left(\frac{\pi}{2}\right)\mathbf{R_X}\left(-\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{4}}\mathbf{I}\right) \qquad \mathbf{S}^\dagger = \mathbf{R_X}\left(\frac{\pi}{2}\right)\mathbf{R_Y}\left(-\frac{\pi}{2}\right)\mathbf{R_X}\left(-\frac{\pi}{2}\right)\left(e^{-i\frac{\pi}{4}}\mathbf{I}\right)$$

$$\mathbf{S} = \mathbf{R_Y}\left(-\frac{\pi}{2}\right)\mathbf{R_X}\left(\frac{\pi}{2}\right)\mathbf{R_Y}\left(\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{4}}\mathbf{I}\right) \qquad \mathbf{S}^\dagger = \mathbf{R_Y}\left(-\frac{\pi}{2}\right)\mathbf{R_X}\left(-\frac{\pi}{2}\right)\mathbf{R_Y}\left(\frac{\pi}{2}\right)\left(e^{-i\frac{\pi}{4}}\mathbf{I}\right)$$

Euler Elemental Rotation Decomposition Examples for $\pi/8$ Gates

- Global Phase Shifts of δ are Represented as $(e^{i\delta}\mathbf{I})$

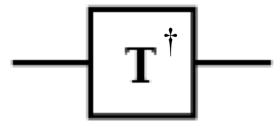


$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i(\pi/4)} \end{bmatrix}$$

$$\mathbf{T} = \mathbf{R}_{\mathbf{Z}}\left(\frac{\pi}{4}\right)\left(e^{i\frac{\pi}{8}}\mathbf{I}\right)$$

$$\mathbf{T} = \mathbf{R}_{\mathbf{X}}\left(\frac{\pi}{2}\right)\mathbf{R}_{\mathbf{Y}}\left(\frac{\pi}{4}\right)\mathbf{R}_{\mathbf{X}}\left(-\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{8}}\mathbf{I}\right)$$

$$\mathbf{T} = \mathbf{R}_{\mathbf{Y}}\left(-\frac{\pi}{2}\right)\mathbf{R}_{\mathbf{X}}\left(\frac{\pi}{4}\right)\mathbf{R}_{\mathbf{Y}}\left(\frac{\pi}{2}\right)\left(e^{i\frac{\pi}{8}}\mathbf{I}\right)$$



$$\mathbf{T}^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i(\pi/4)} \end{bmatrix}$$

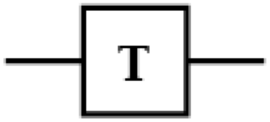
$$\mathbf{T}^{\dagger} = \mathbf{R}_{\mathbf{Z}}\left(-\frac{\pi}{4}\right)\left(e^{-i\frac{\pi}{8}}\mathbf{I}\right)$$

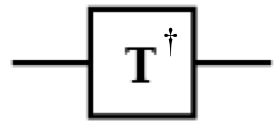
$$\mathbf{T}^{\dagger} = \mathbf{R}_{\mathbf{X}}\left(\frac{\pi}{2}\right)\mathbf{R}_{\mathbf{Y}}\left(-\frac{\pi}{4}\right)\mathbf{R}_{\mathbf{X}}\left(-\frac{\pi}{2}\right)\left(e^{-i\frac{\pi}{8}}\mathbf{I}\right)$$

$$\mathbf{T}^{\dagger} = \mathbf{R}_{\mathbf{Y}}\left(-\frac{\pi}{2}\right)\mathbf{R}_{\mathbf{X}}\left(-\frac{\pi}{4}\right)\mathbf{R}_{\mathbf{Y}}\left(\frac{\pi}{2}\right)\left(e^{-i\frac{\pi}{8}}\mathbf{I}\right)$$

Euler Elemental Rotation Decomposition Examples for $\pi/8$ Gates

- Global Phase Shifts of δ are Represented as $(e^{i\delta}\mathbf{I})$





$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i(\pi/4)} \end{bmatrix} \qquad \mathbf{T}^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i(\pi/4)} \end{bmatrix}$$

$$\mathbf{T}^{\dagger} = \mathbf{SSST} = \mathbf{SZT} \qquad \mathbf{SS} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \mathbf{Z}$$

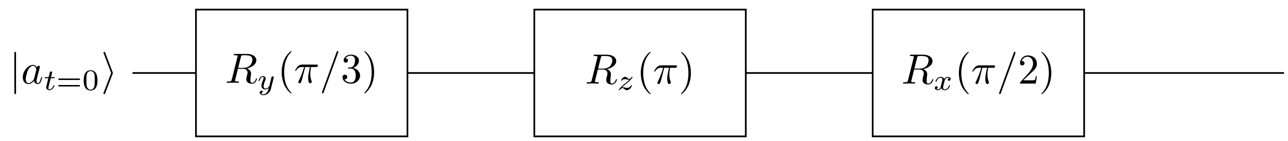
$$\mathbf{T}^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -e^{i\frac{\pi}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -ie^{i\frac{\pi}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\frac{\pi}{2}}e^{i\frac{\pi}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{bmatrix}$$

$$\mathbf{T} = \mathbf{SSST}^{\dagger} = \mathbf{SZT}^{\dagger} \qquad \mathbf{SS} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \mathbf{Z}$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -e^{-i\frac{\pi}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -ie^{-i\frac{\pi}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\frac{\pi}{2}}e^{-i\frac{\pi}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\frac{3\pi}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$$

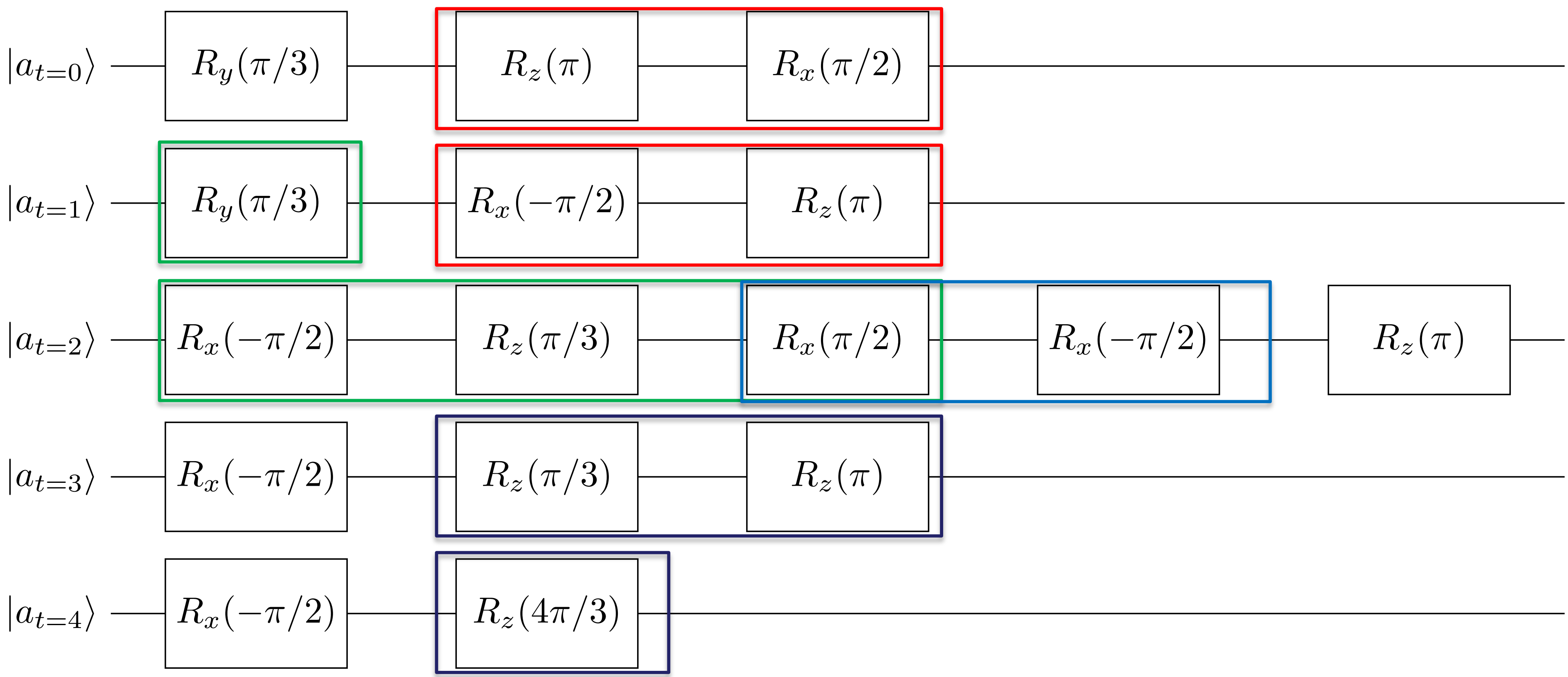
Example: Qubit Operator Optimization

Goal: Maximize $\mathbf{R_Z}(\delta)$ Rotations:



Key: **Step 1**
Step 2
Step 3
Step 4

Optimization Steps:

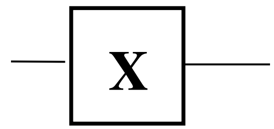
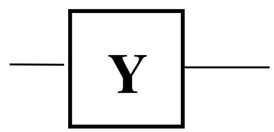
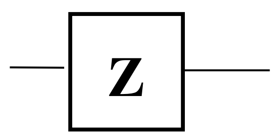


Single Qubit General Rotations

$\mathbf{R}_x(\theta)$		$\begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$
$\mathbf{R}_y(\theta)$		$\begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$
$\mathbf{R}_z(\theta)$		$\begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$

Single Qubit Pauli Operators

(180-degree rotations about axes in Bloch Sphere)

Pauli-**X**		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli-**Y**		$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Pauli-**Z** (π/2 gate)		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

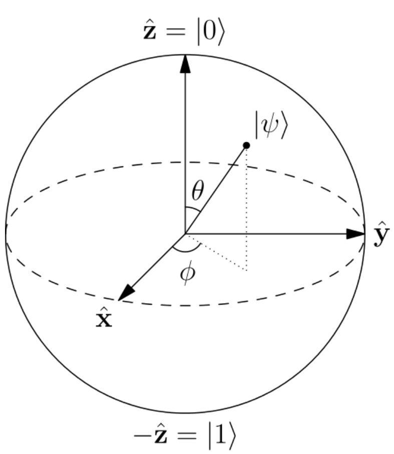
Other Single Qubit Operations

Hadamard	H	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Phase (π/4 gate)	S	$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
“T” (π/8 gate)	T	$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

Completely Generalized Rotation

$$0 \leq \theta \leq \pi$$

$$0 \leq \phi \leq 2\pi$$



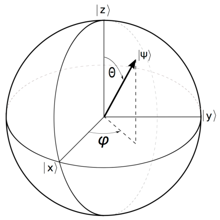
$$\mathbf{R}(\theta,\phi) = |\psi\rangle\langle\psi| = \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix} \otimes \begin{bmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+\cos\theta & \cos\phi\sin\theta - i\sin\phi\sin\theta \\ \cos\phi\sin\theta + i\sin\phi\cos\theta & 1-\cos\theta \end{bmatrix}$$

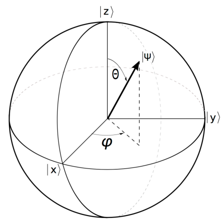
Binary Basis States (Eigenstates)

Photonic Example



Pauli Gate	Optical Name	Basis State (Jones Vector)	Bloch Sphere
$\sigma_1 = \sigma_X = \mathbf{X}$	Diag./Anti-dia. Slant-45 X-Basis	$\left\|+\right\rangle = \left(\left\|0\right\rangle + \left\|1\right\rangle\right)/\sqrt{2}$	$\hat{\mathbf{x}}$
		$\left\|-\right\rangle = \left(\left\|0\right\rangle - \left\|1\right\rangle\right)/\sqrt{2}$	$-\hat{\mathbf{x}}$
$\sigma_2 = \sigma_Y = \mathbf{Y}$	LHP/RHP Circular Y-Basis	$\left\|+i\right\rangle = \left(\left\|0\right\rangle + i\left\|1\right\rangle\right)/\sqrt{2}$	$\hat{\mathbf{y}}$
		$\left\|-i\right\rangle = \left(\left\|0\right\rangle - i\left\|1\right\rangle\right)/\sqrt{2}$	$-\hat{\mathbf{y}}$
$\sigma_3 = \sigma_Z = \mathbf{Z}$	Horizontal/ Vertical Computational **Z**-basis	$\left\|0\right\rangle$	$\hat{\mathbf{z}}$
		$\left\|1\right\rangle$	$-\hat{\mathbf{z}}$

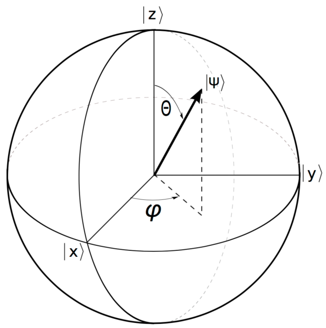
Binary Basis States (Eigenstates)
alternative notation



$$\left|L\right\rangle = \left(\left|H\right\rangle + i\left|V\right\rangle\right)\Big/\sqrt{2} = \left|\circlearrowleft\right\rangle \qquad \left|R\right\rangle = \left(\left|H\right\rangle + i\left|V\right\rangle\right)\Big/\sqrt{2} = \left|\circlearrowright\right\rangle$$

Pauli Gate	Optical Name	Basis State (Jones Vector)	Jones Vector (standard)
$\sigma_1 = \sigma_X = \mathbf{X}$	Diag./Anti-dia. Slant-45 X-Basis	$\left\|D\right\rangle = \left(\left\|H\right\rangle + \left\|V\right\rangle\right)\Big/\sqrt{2}$	$\hat{\mathbf{x}} = \frac{1}{\sqrt{2}}\left[\begin{matrix} 1 & 1 \end{matrix}\right]^{\mathrm{T}}$
		$\left\|A\right\rangle = \left(\left\|H\right\rangle - \left\|V\right\rangle\right)\Big/\sqrt{2}$	$-\hat{\mathbf{x}} = \frac{1}{\sqrt{2}}\left[\begin{matrix} 1 & -1 \end{matrix}\right]^{\mathbf{T}}$
$\sigma_2 = \sigma_Y = \mathbf{Y}$	LHP/RHP Circular Y-Basis	$\left\|L\right\rangle = \left(\left\|H\right\rangle + i\left\|V\right\rangle\right)\Big/\sqrt{2}$	$\hat{\mathbf{y}} = \frac{1}{\sqrt{2}}\left[\begin{matrix} 1 & i \end{matrix}\right]^{\mathrm{T}}$
		$\left\|R\right\rangle = \left(\left\|H\right\rangle - i\left\|V\right\rangle\right)\Big/\sqrt{2}$	$-\hat{\mathbf{y}} = \frac{1}{\sqrt{2}}\left[\begin{matrix} 1 & -i \end{matrix}\right]^{\mathrm{T}}$
$\sigma_3 = \sigma_Z = \mathbf{Z}$	Horizontal/Vertical Computational Z-basis	$\left\|0\right\rangle = \left\|H\right\rangle$	$\hat{\mathbf{z}} = \left[\begin{matrix} 1 & 0 \end{matrix}\right]^{\mathrm{T}}$
		$\left\|1\right\rangle = \left\|V\right\rangle$	$-\hat{\mathbf{z}} = \left[\begin{matrix} 0 & 1 \end{matrix}\right]^{\mathrm{T}}$

Single Qubit Operator Relationships



Pauli Gate	Eigenvalue	Eigenvector	
$\sigma_1 = \sigma_\mathbf{X} = \mathbf{X}$	+1	$\lvert + \rangle = \left(\lvert 0 \rangle + \lvert 1 \rangle \right) / \sqrt{2}$	(Diagonal)
$\sigma_1 = \sigma_\mathbf{X} = \mathbf{X}$	-1	$\lvert - \rangle = \left(\lvert 0 \rangle - \lvert 1 \rangle \right) / \sqrt{2}$	(Anti-Diagonal)
$\sigma_2 = \sigma_\mathbf{Y} = \mathbf{Y}$	+1	$\lvert +i \rangle = \left(\lvert 0 \rangle + i \lvert 1 \rangle \right) / \sqrt{2}$	
$\sigma_2 = \sigma_\mathbf{Y} = \mathbf{Y}$	-1	$\lvert -i \rangle = \left(\lvert 0 \rangle - i \lvert 1 \rangle \right) / \sqrt{2}$	
$\sigma_3 = \sigma_\mathbf{Z} = \mathbf{Z}$	+1	$\lvert 0 \rangle$	
$\sigma_3 = \sigma_\mathbf{Z} = \mathbf{Z}$	-1	$\lvert 1 \rangle$	

Quantum Logic Gates and Circuits

Multi-qubit “Gates” or Operators

Multi Qubit Systems (Circuits)

- Multi qubit systems are represented in terms of a “product quantum state”
- Consider a System of Two qubits, the state of this system is a superposition of:

$$|pq\rangle = \alpha|00\rangle + \beta|00\rangle + \chi|00\rangle + \delta|00\rangle$$

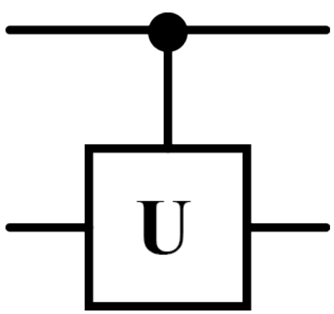
$$\begin{bmatrix} \alpha \\ \beta \\ \chi \\ \delta \end{bmatrix} \begin{matrix} \longleftarrow \text{Amplitude for 00} \\ \longleftarrow \text{Amplitude for 01} \\ \longleftarrow \text{Amplitude for 10} \\ \longleftarrow \text{Amplitude for 11} \end{matrix}$$

2-Qubit (Controlled) Gates

- General Model where Top Qubit Allows Bottom Qubit to Evolve with **U** (or not)
- Recall that:

$$\mathbf{U} = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$$

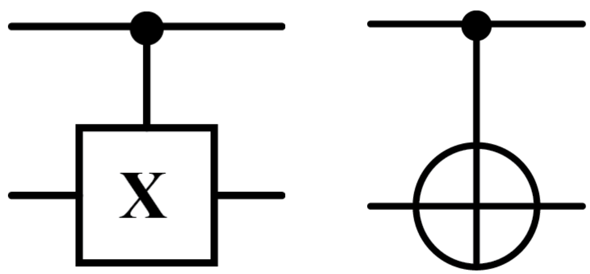
- Controlled-**U** Gate



$$\mathbf{C_U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & u_{10} & u_{11} \end{bmatrix}$$

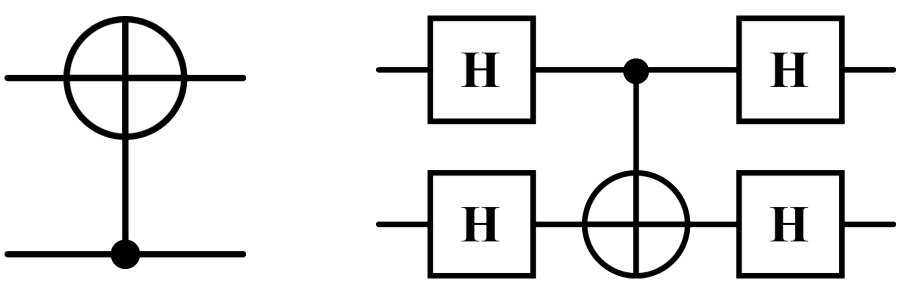
Two-Qubit Gates

- Bloch Sphere can only Represent a Single Qubit
- Feynman, Controlled-NOT, CNOT, Controlled-**X**



$$\mathbf{CNOT} = \mathbf{C_X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

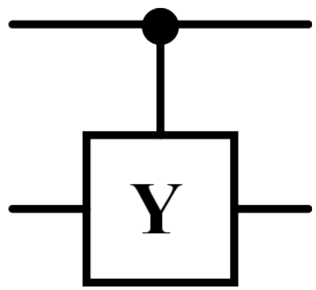
- Inverted SWAP, Inverted Controlled-**X**, (technically not a different gate)



$$\mathbf{ISWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

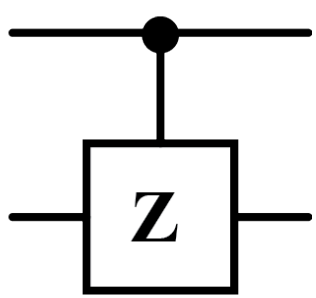
Two-Qubit Gates (cont)

- Controlled-**Y**



$$\mathbf{C_Y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$$

- Controlled-**Z**, Controlled-Sign, Controlled-Phase, **CPHASE**, **CSIGN**
  - arises in linear optical computing



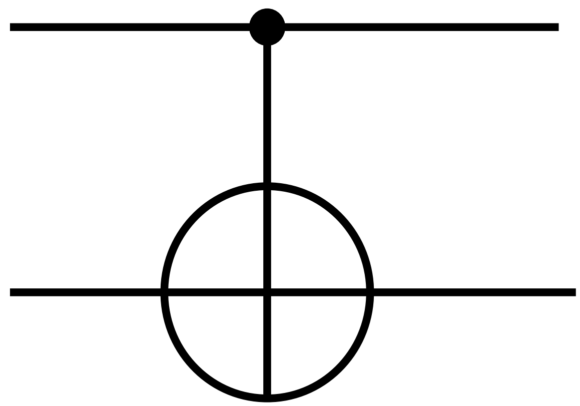
$$\mathbf{C_Z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Controlled-NOT (C_{NOT}) Gate

(*aka* Feynman, Controlled-$\mathbf{X}$, Quantum XOR Gate)

$|a_1\rangle$ **Control Input**

$|a_2\rangle$ **Target Input**



$$|b_1\rangle = |a_1\rangle$$

$$|b_2\rangle = |a_1 \oplus a_2\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\|a_1\rangle$	$\|a_2\rangle$	$\|b_1\rangle$	$\|b_2\rangle$
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

Controlled-NOT ($\mathbf{C}_{\mathrm{NOT}}$) Gate
(*aka* Feynman, Controlled-$\mathbf{X}$, Quantum XOR Gate)

$$\mathbf{C}_{NOT} = \mathbf{C}_{\mathbf{X}} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$$

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\langle 00| = \langle 0| \otimes \langle 0| = \begin{bmatrix} 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\langle 01| = \langle 0| \otimes \langle 1| = \begin{bmatrix} 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

Controlled-NOT ($\mathbf{C}_{\mathrm{NOT}}$) Gate

(*aka* Feynman, Controlled-$\mathbf{X}$, Quantum XOR Gate)

$$\mathbf{C}_{\mathbf{X}} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$$

$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\langle 10| = \langle 1| \otimes \langle 0| = \begin{bmatrix} 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

$$|11\rangle = |1\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\langle 11| = \langle 1| \otimes \langle 1| = \begin{bmatrix} 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

Controlled-NOT ($\mathbf{C}_{\mathrm{NOT}}$) Gate
(*aka* Feynman, Controlled-$\mathbf{X}$, Quantum XOR Gate)

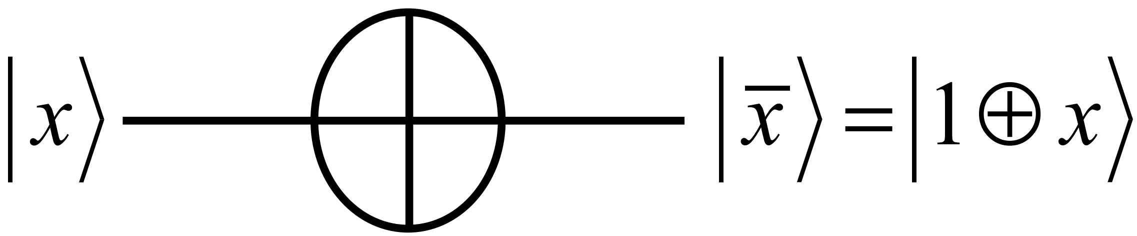
$$\mathbf{C_X} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$$

$$\mathbf{C_X} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

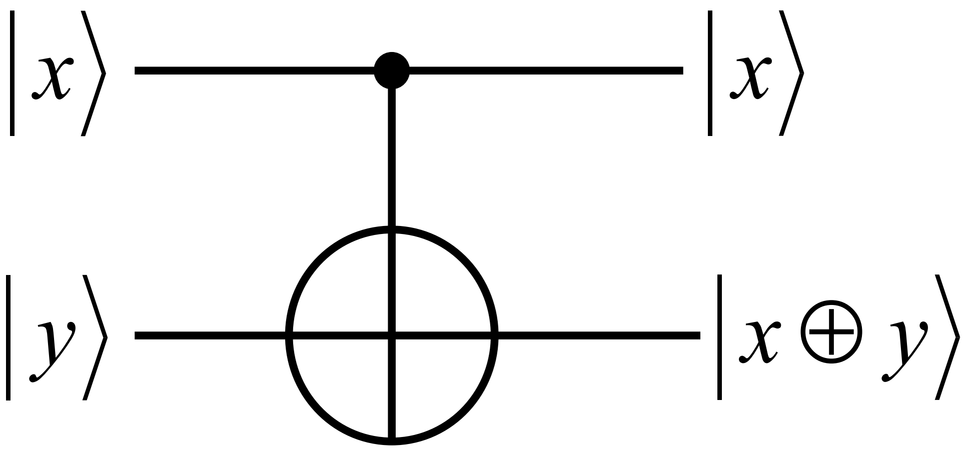
$$+ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{C_X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\mathbf{X}$ and $\mathbf{C_X}$ Gates
(*aka* Pauli-$\mathbf{X}$/NOT and Controlled-NOT)

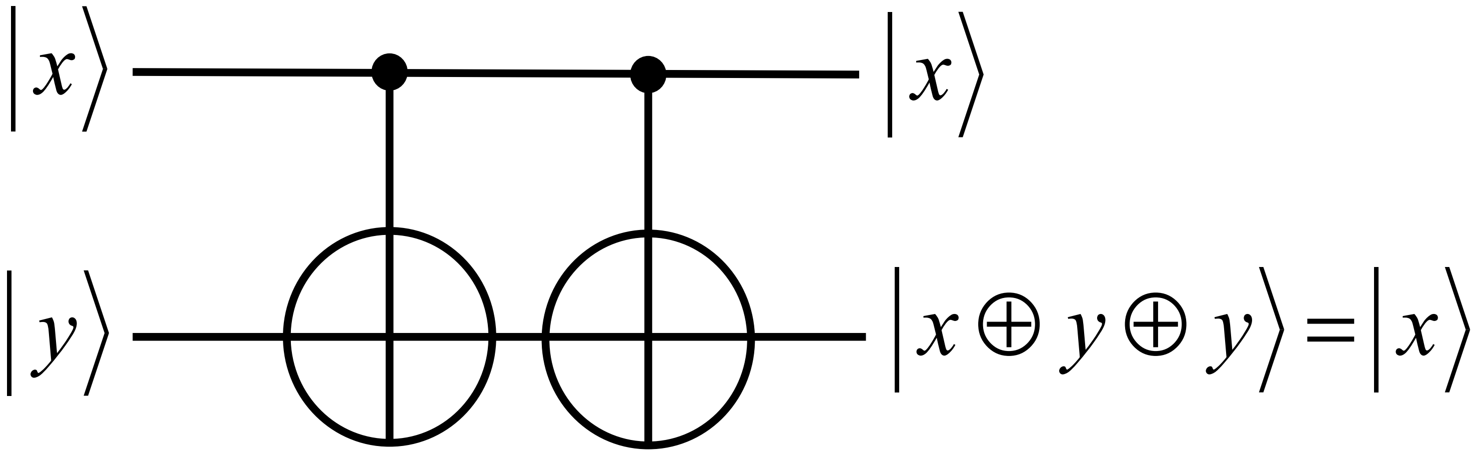


$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

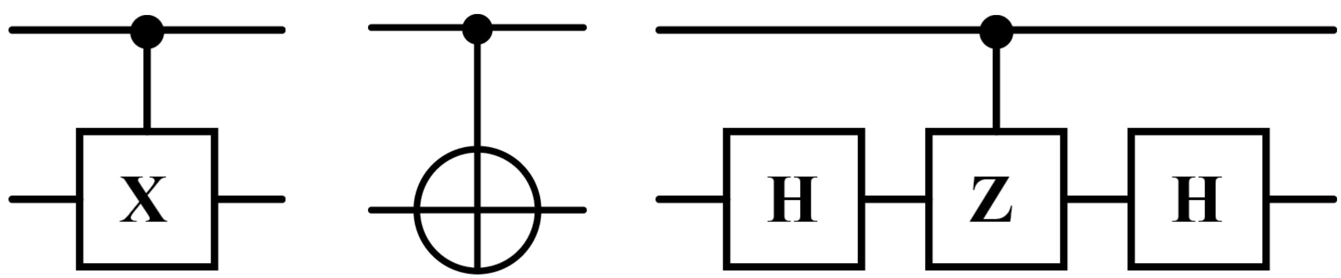
Reversibility of C_X Gate



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

CNOT Using CPHASE and H

- Can Build **CNOT** with **CPHASE** and **H**



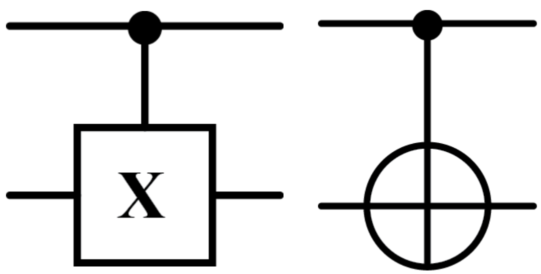
$$\mathbf{CNOT} = \mathbf{C_X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$(\mathbf{I} \otimes \mathbf{H})\mathbf{C_Z}(\mathbf{I} \otimes \mathbf{H}) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

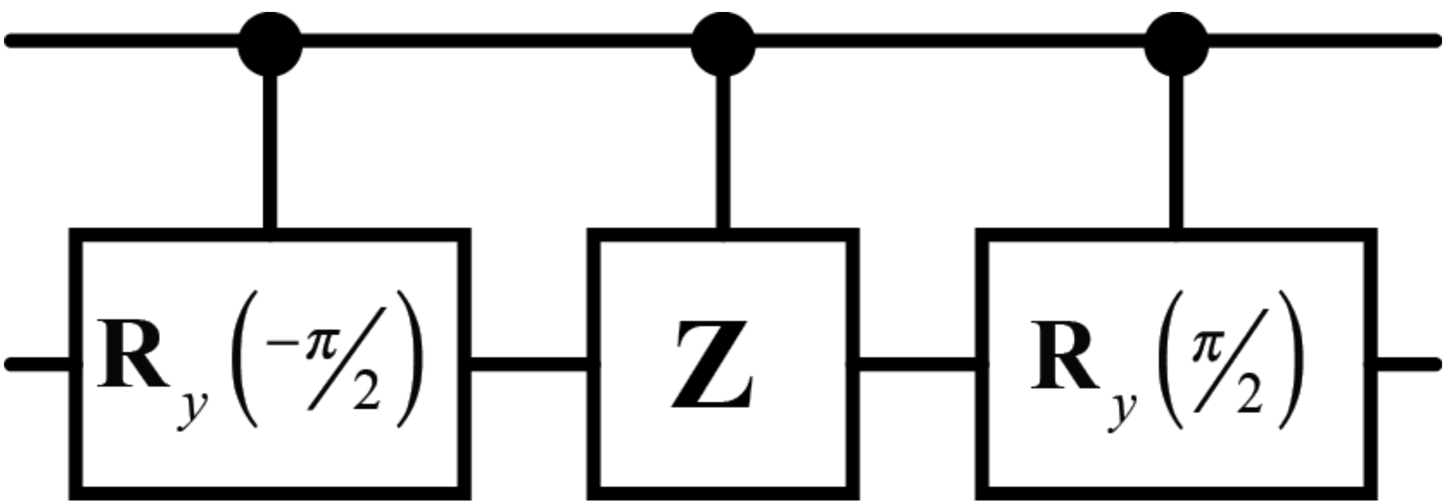
$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

CNOT Using **CPHASE** and $\mathbf{R}_y$

- Can Build **CNOT** with **CPHASE** and $\mathbf{R}_y$

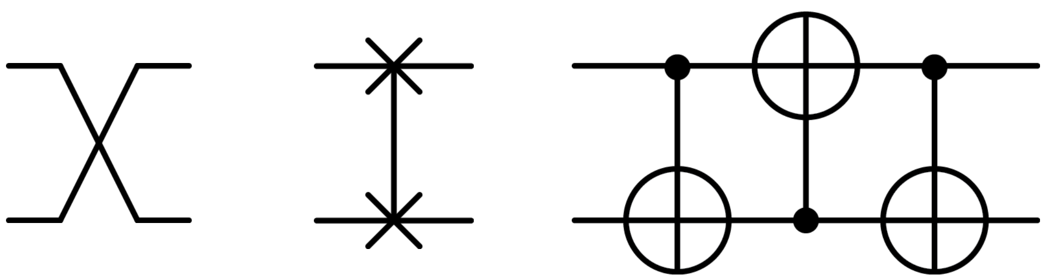


$$\mathbf{CNOT} = \mathbf{C}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



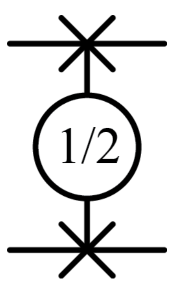
Two-Qubit Gates (cont)

- $SWAP$ Gate



$$\mathbf{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

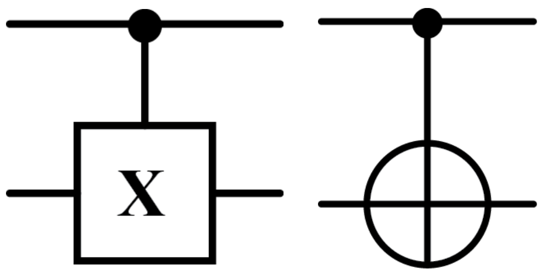
- Square Root of $\mathbf{SWAP}$, $\sqrt{\mathbf{SWAP}}$
  - spintronic-based circuits (“exchange interaction”)



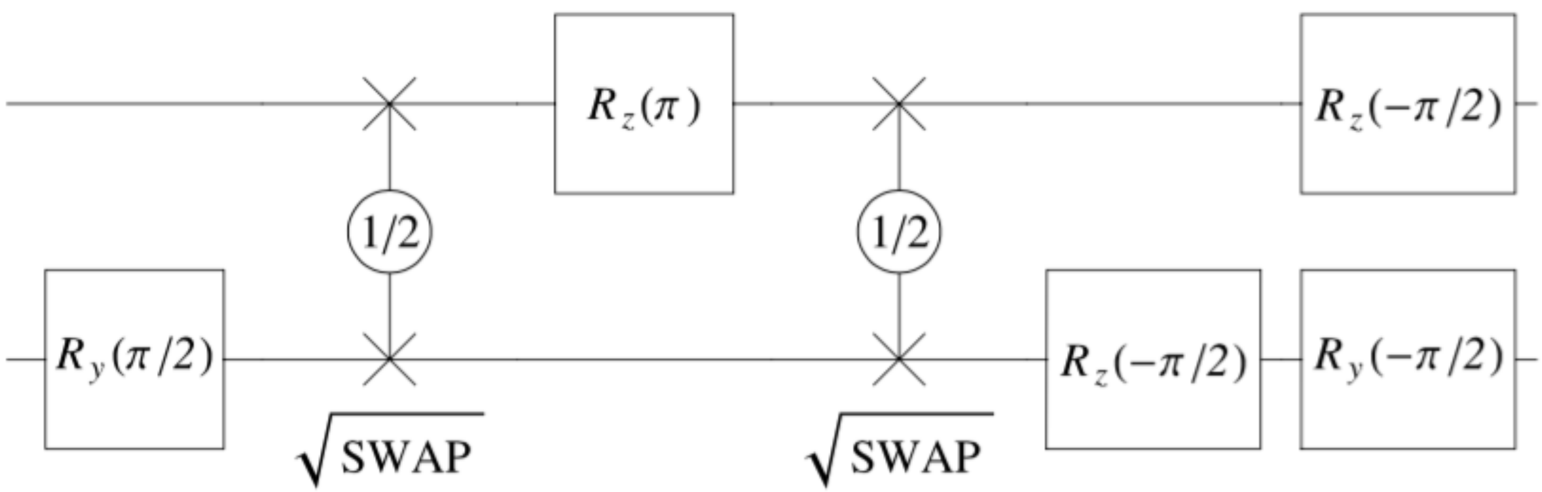
$$\mathbf{Q}_{\mathbf{SWP}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \tfrac{1}{2}(1+i) & \tfrac{1}{2}(1-i) & 0 \\ 0 & \tfrac{1}{2}(1-i) & \tfrac{1}{2}(1+i) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

CNOT Using **CPHASE** and Sq. root **SWAP**, $\mathbf{R_y}$, and $\mathbf{R_z}$

- Can Build **CNOT** with **CPHASE**, Sq. root **SWAP**, $\mathbf{R_y}$ and $\mathbf{R_z}$



$$\mathbf{CNOT} = \mathbf{C_x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



Two-Qubit Gates (cont)

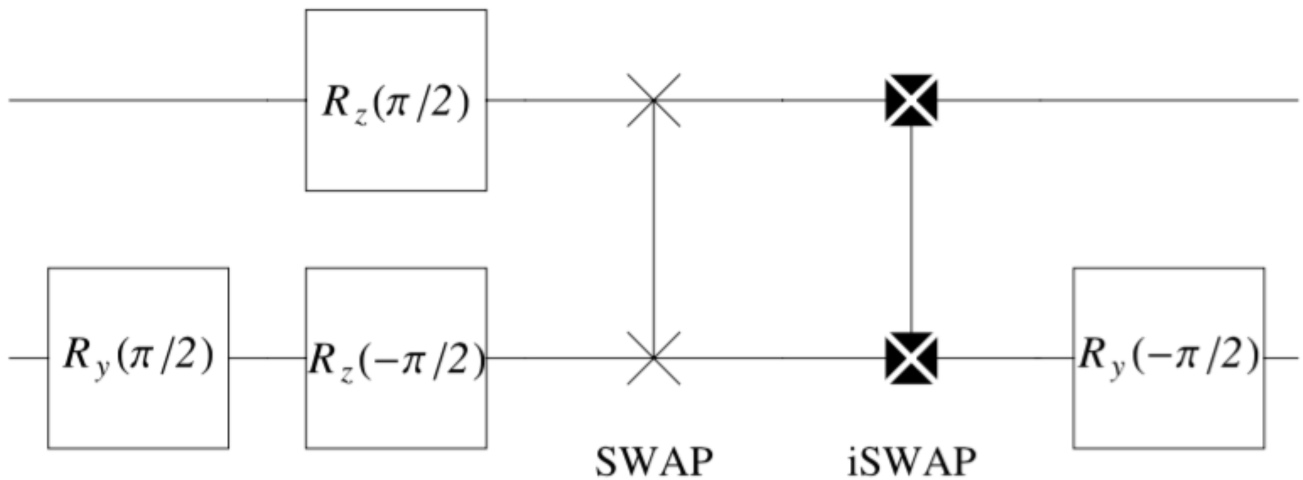
- ***i*SWAP** Gate
  - arises in superconducting quantum computing with Hamiltonians that implement the so-called **XY** Model

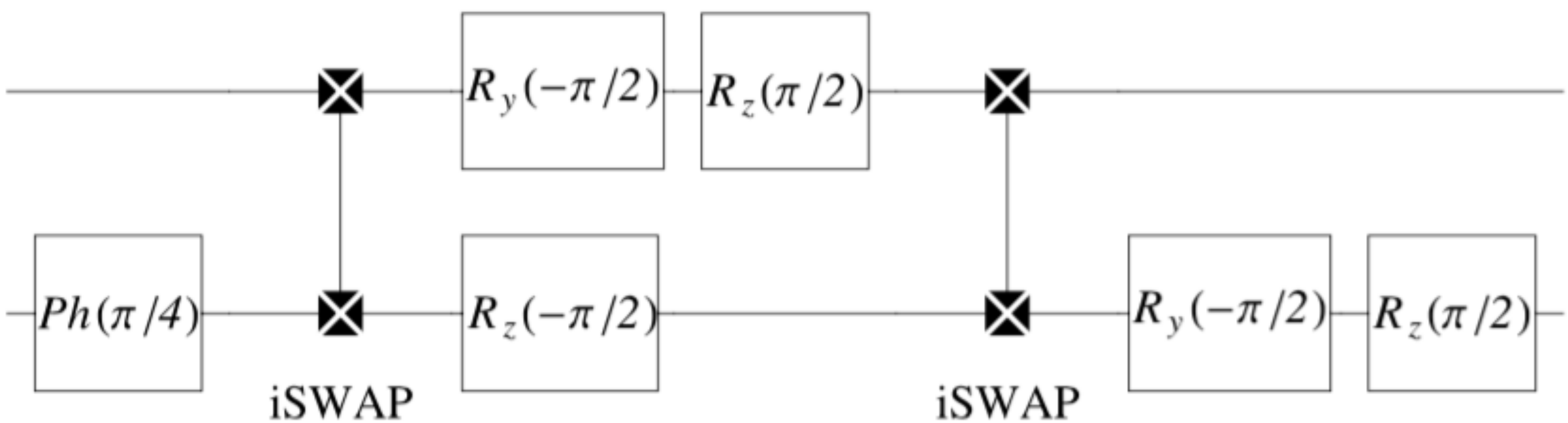
$$i\mathbf{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Square Root of $\mathbf{SWAP}^{\alpha}$
  - spintronic-based circuits (“exchange interaction”)
  - duration of exchange determines the exponent

$$\mathbf{SWAP}^{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \tfrac{1}{2}\left(1+e^{i\pi\alpha}\right) & \tfrac{1}{2}\left(1-e^{i\pi\alpha}\right) & 0 \\ 0 & \tfrac{1}{2}\left(1-e^{i\pi\alpha}\right) & \tfrac{1}{2}\left(1+e^{i\pi\alpha}\right) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

CNOT Using **CPHASE** and Sq. root **SWAP**, *i***SWAP, R**$_y$, and **R**$_z$





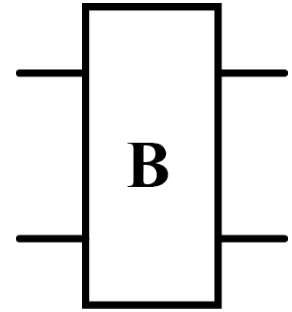
Two-Qubit Gates (cont)

- **BARENCO** Gate
  - ϕ, α, and θ are fixed irrational multiples of each and with π
- In Form of a Controlled-**U** Gate

$$\mathbf{BARENCO} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\alpha}\cos(\theta) & -ie^{i(\alpha-\phi)}\sin(\theta) \\ 0 & 0 & -ie^{i(\alpha-\phi)}\sin(\theta) & e^{i\alpha}\cos(\theta) \end{bmatrix}$$

Berkeley **B** Gate

- Hamiltonian is: $\mathcal{H} = \frac{\pi}{8}\left(2\mathbf{X}\otimes\mathbf{X} + \mathbf{Y}\otimes\mathbf{Y}\right)$
- Transfer Matrix: $\mathbf{B} = e^{i\mathcal{H}} = e^{i\frac{\pi}{8}\left(2\mathbf{X}\otimes\mathbf{X}+\mathbf{Y}\otimes\mathbf{Y}\right)}$



$$\mathbf{B} = \begin{bmatrix} \cos\left(\frac{\pi}{8}\right) & 0 & 0 & i\sin\left(\frac{\pi}{8}\right) \\ 0 & \cos\left(\frac{3\pi}{8}\right) & i\sin\left(\frac{3\pi}{8}\right) & 0 \\ 0 & i\sin\left(\frac{3\pi}{8}\right) & \cos\left(\frac{3\pi}{8}\right) & 0 \\ i\sin\left(\frac{\pi}{8}\right) & 0 & 0 & \cos\left(\frac{\pi}{8}\right) \end{bmatrix} = \frac{\sqrt{2-\sqrt{2}}}{2}\begin{bmatrix} 1+\sqrt{2} & 0 & 0 & i \\ 0 & 1 & i\left(1+\sqrt{2}\right) & 0 \\ 0 & i\left(1+\sqrt{2}\right) & 1 & 0 \\ i & 0 & 0 & 1+\sqrt{2} \end{bmatrix}$$

Quantum Logic Gates and Circuits

Three-qubit Gates

Three-Qubit Gates

- Toffoli Gate $\mathbf{Toff}\left|a,b,c\right\rangle = \left|a,b,ab\oplus c\right\rangle$
  - universal for reversible with Pauli-X and CNOT
- Peres Gate $\mathbf{P}\left|a,b,c\right\rangle = \left|a,a\oplus b,ab\oplus c\right\rangle$
- Fredkin (Controlled-SWAP)
- Deutsch Gate
  - universal

Deutsch Gate

- θ is any constant angle, such that $\frac{2\theta}{\pi}$ is an irrational number.

$$\textbf{DEUTSCH} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\cos(\theta) & \sin(\theta) \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin(\theta) & i\cos(\theta) \end{bmatrix}$$

Generalized Toffoli Gate

- Generalized Toffoli Gate
  - Some Papers Refer to Generalized Toffoli as Toffoli
- NOT (Pauli-X), CNOT (Feynman), Toffoli are Universal Set of Reversible Logic Gates
  - Reversible Logic assumes Only Basis States are Used
  - Form of Classical Switching Theory Based Computation (e.g., electronic adiabatic circuits)

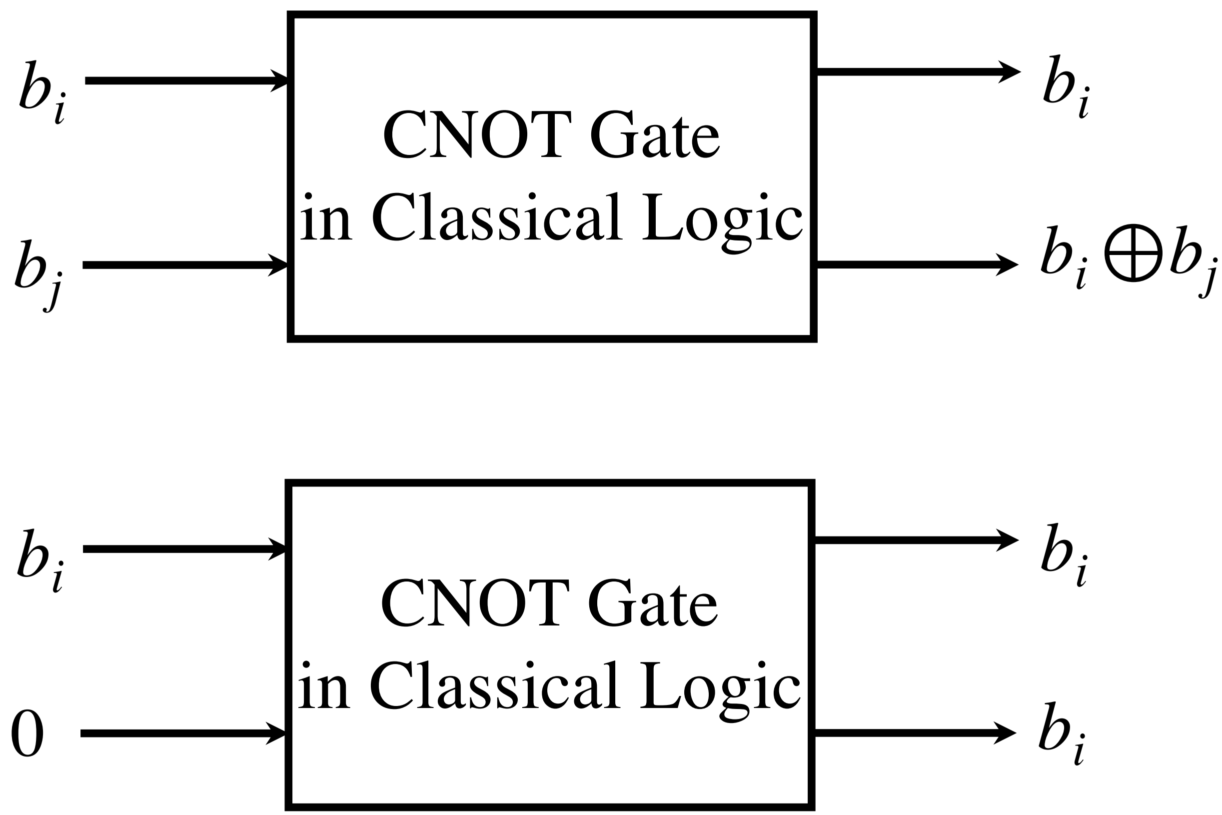
Decompositions: Barenco's Theorem

- All Multi-qubit Gates can be Decomposed into a Cascade of Two-qubit Controlled and Single Qubit Rotations
- "Quantum Cost" is the Total Number of Single and Two-qubit Controlled Gates Required to Perform a Computation
- Other Decompositions are Known
- New Decompositions are an Open Area of Research

Quantum Logic Gates and Circuits

The “No-Cloning” Theorem

Bit Copying

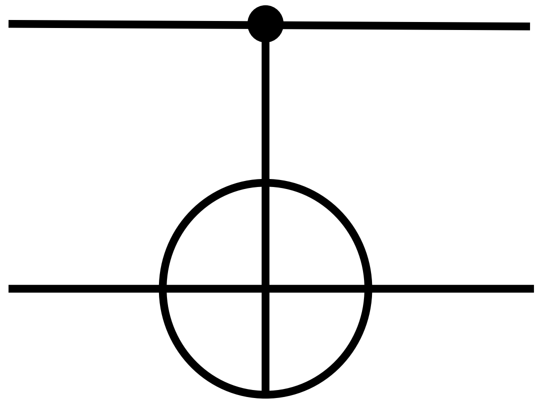


Can Produce a Fanout (copy) with CNOT Gate implemented in CLASSICAL Logic

Is Qubit Copying Possible?

$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$

?



$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$

$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$

- If ***<u>Qubit Copying Possible</u>***, Output Quantum State is:

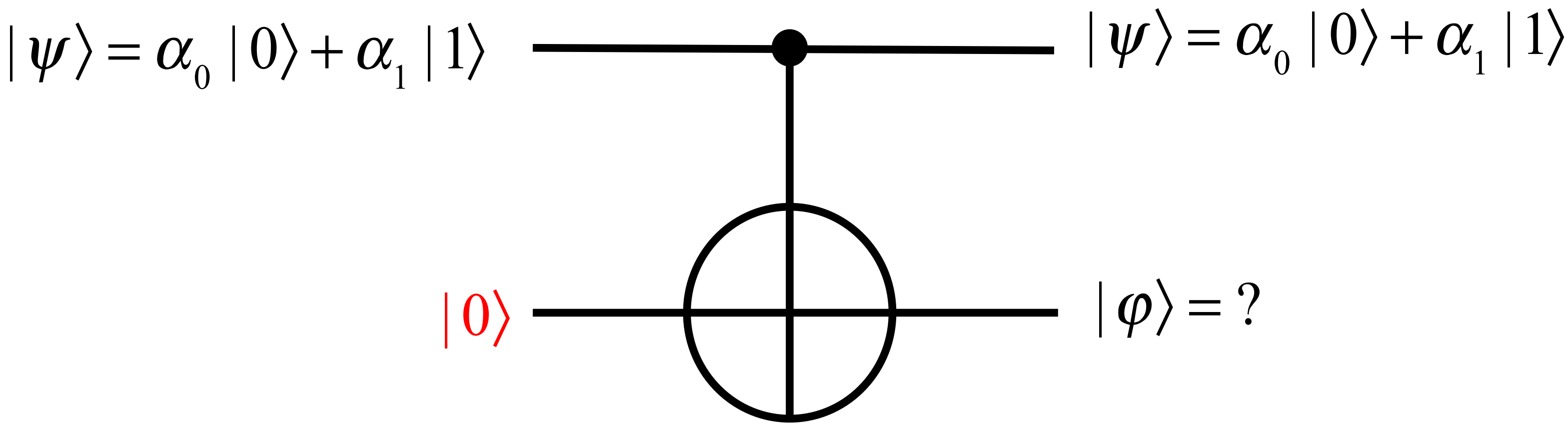
$$|\psi\rangle \otimes |\psi\rangle = |\psi\psi\rangle = (\alpha_0 |0\rangle + \alpha_1 |1\rangle)(\alpha_0 |0\rangle + \alpha_1 |1\rangle)$$

$$= |\psi\psi\rangle = \alpha_0^2 |00\rangle + \alpha_0\alpha_1 |01\rangle + \alpha_0\alpha_1 |10\rangle + \alpha_1^2 |11\rangle$$

- Or Using Tensor Product:

$$|\psi\rangle \otimes |\psi\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \otimes \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_0^2 \\ \alpha_0\alpha_1 \\ \alpha_0\alpha_1 \\ \alpha_1^2 \end{bmatrix}$$

Qubit Copying?



- Input Quantum State:

$$|\psi\rangle \otimes |0\rangle = |\psi 0\rangle = (\alpha_0 |0\rangle + \alpha_1 |1\rangle)(|0\rangle) = \alpha_0 |00\rangle + \alpha_1 |10\rangle$$

- Output Quantum State:

$$|\psi\varphi\rangle = \mathbf{CNOT} |\psi 0\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ 0 \\ \alpha_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ 0 \\ 0 \\ \alpha_1 \end{bmatrix} = \alpha_0 |00\rangle + \alpha_1 |11\rangle$$

- Qubit is ***NOT COPIED***:

$$\alpha_0 |00\rangle + \alpha_1 |11\rangle \neq \alpha_0^2 |00\rangle + \alpha_0\alpha_1 |01\rangle + \alpha_0\alpha_1 |10\rangle + \alpha_1^2 |11\rangle$$

No-Cloning Theorem

- Transformations Carried out by Quantum Gates are Unitary
- Cloning of a Quantum State is a Non-Unitary and Non-Linear Process
- Information Point of View is Two Copies of Quantum State Embody MORE Information than Available in One Copy
- IS POSSIBLE to Clone States After a Measurement has Occurred
- No Cloning Theorem Applies to UNKNOWN Quantum States

No-Cloning Theorem

- Proof by Contradiction
- Assume a Cloning Gate Exists Characterized by Transform Matrix $\mathbf{G}$
- Assume Two Orthogonal Quantum States are Cloned, One after the Other

$$\mathbf{G}(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle \quad \text{OR} \quad \mathbf{G}(|\psi 0\rangle) = |\psi\psi\rangle$$

$$\mathbf{G}(|\varphi\rangle \otimes |0\rangle) = |\varphi\rangle \otimes |\varphi\rangle \quad \text{OR} \quad \mathbf{G}(|\varphi 0\rangle) = |\varphi\varphi\rangle$$

- These Two Equations State that $\mathbf{G}$ Performs a Cloning Operation When the Second Qubit is ket-zero

No-Cloning Theorem

- Consider Another Quantum State:

$$|\xi\rangle = (1/\sqrt{2})(|\psi\rangle + |\varphi\rangle)$$

- Applying the Cloning Transform:

$$\mathbf{G}(|\xi 0\rangle) = \frac{1}{\sqrt{2}}[\mathbf{G}(|\psi 0\rangle) + \mathbf{G}(|\varphi 0\rangle)] = \frac{1}{\sqrt{2}}[|\psi\psi\rangle + |\varphi\varphi\rangle]$$

- If G is Truly a Cloning Gate then:

$$\mathbf{G}(|\xi 0\rangle) = |\xi\xi\rangle$$

- But:

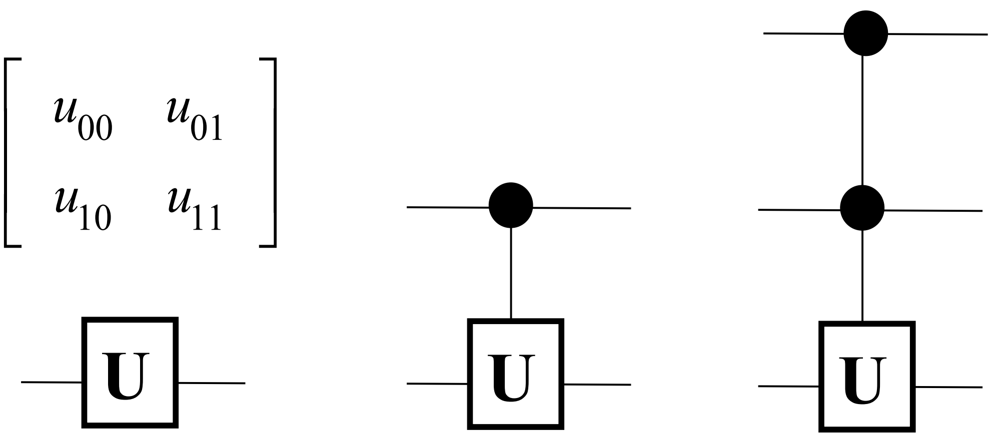
$$|\xi\xi\rangle = \left(\frac{|\psi\rangle + |\varphi\rangle}{\sqrt{2}}\right)\left(\frac{|\psi\rangle + |\varphi\rangle}{\sqrt{2}}\right) = \frac{1}{2}(|\psi\psi\rangle + |\psi\varphi\rangle + |\varphi\psi\rangle + |\varphi\varphi\rangle)$$

CONTRADICTION!!!

Quantum Gates*

- General controlled gates that control some 1-qubit unitary operation **U** are useful

$$\begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$$



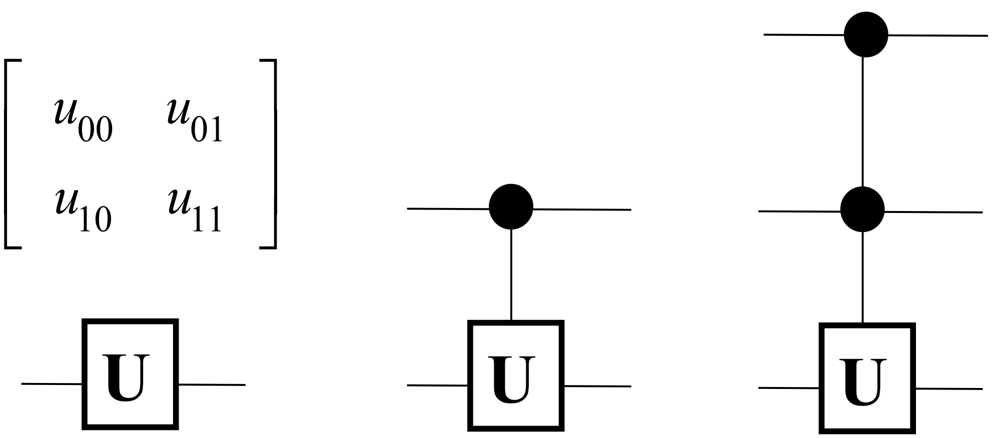
etc.

U   **C(U)**   **C(C(U))=C^2(U)**

Quantum Gates

- General controlled gates that control some 1-qubit unitary operation $\mathbf{U}$ are useful

$$\begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$$



$\mathbf{U}$ $\quad$ $\mathbf{C}(\mathbf{U})$ $\quad$ $\mathbf{C}(\mathbf{C}(\mathbf{U}))=\mathbf{C}^2(\mathbf{U})$

$$\mathbf{C}(\mathbf{U}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & u_{10} & u_{11} \end{bmatrix}$$

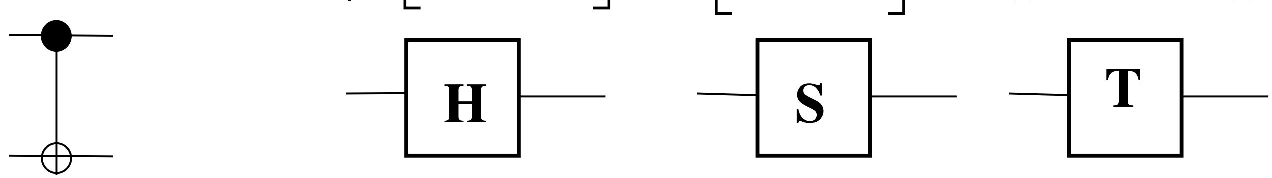
$$\mathbf{C}^2(\mathbf{U}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & 0 & 0 & 0 & 0 & u_{10} & u_{11} \end{bmatrix}$$

Quantum Gates*

Discrete Universal Gate Set Example

- **Example 1**: Four-member "standard" gate set, $\{\mathbf{C_X}, \mathbf{H}, \mathbf{S}, \mathbf{T}\}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = e^{i\pi/8}\begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{bmatrix}$$



$\mathbf{C_X}$ Hadamard Phase π/8 (**T**) gate

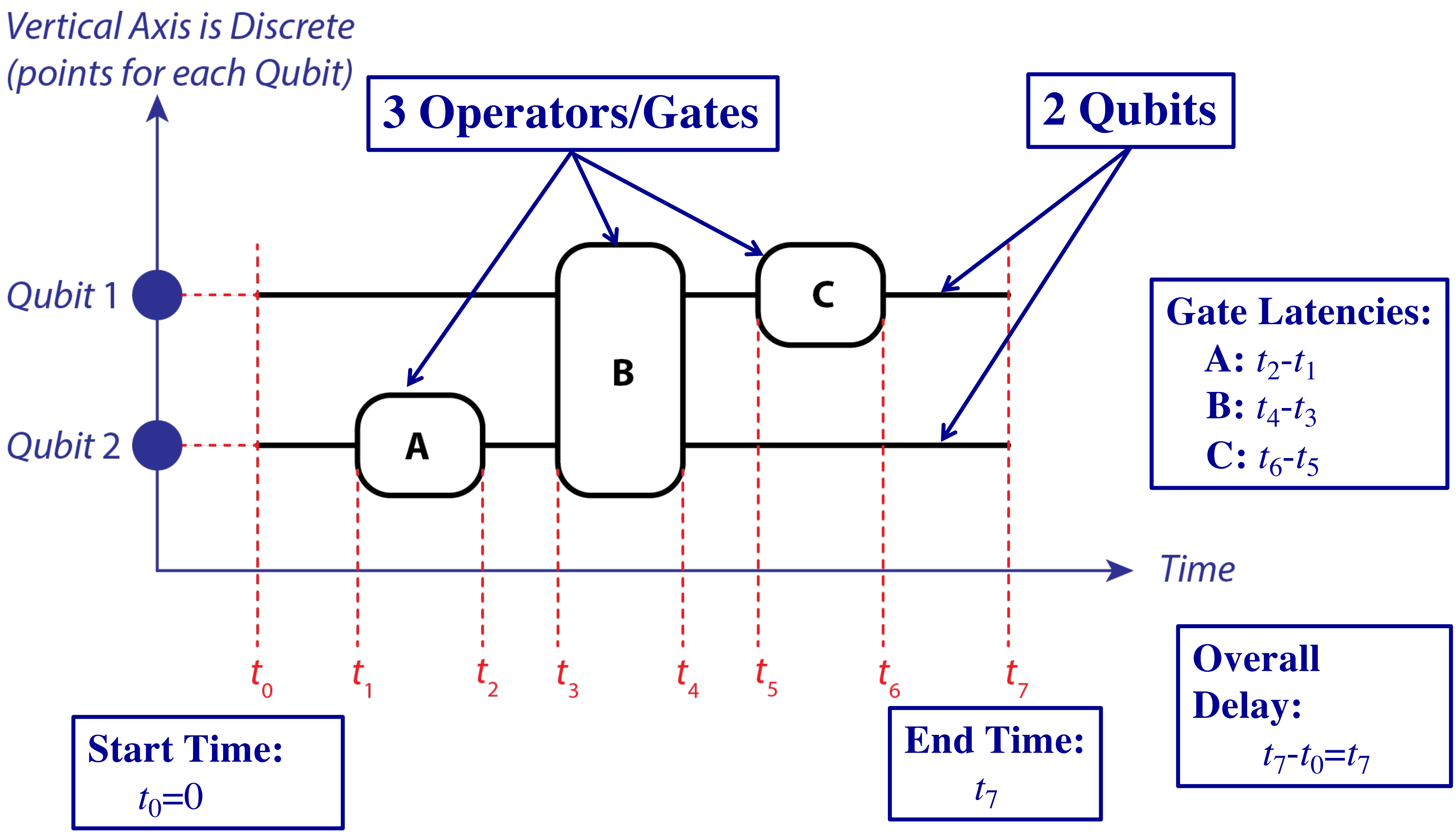
- **Example** 2: $\{\mathbf{X}, \mathbf{C_X}, \mathbf{H}$, Toffoli$\}$

Quantum Logic Gates and Circuits

Quantum Programs

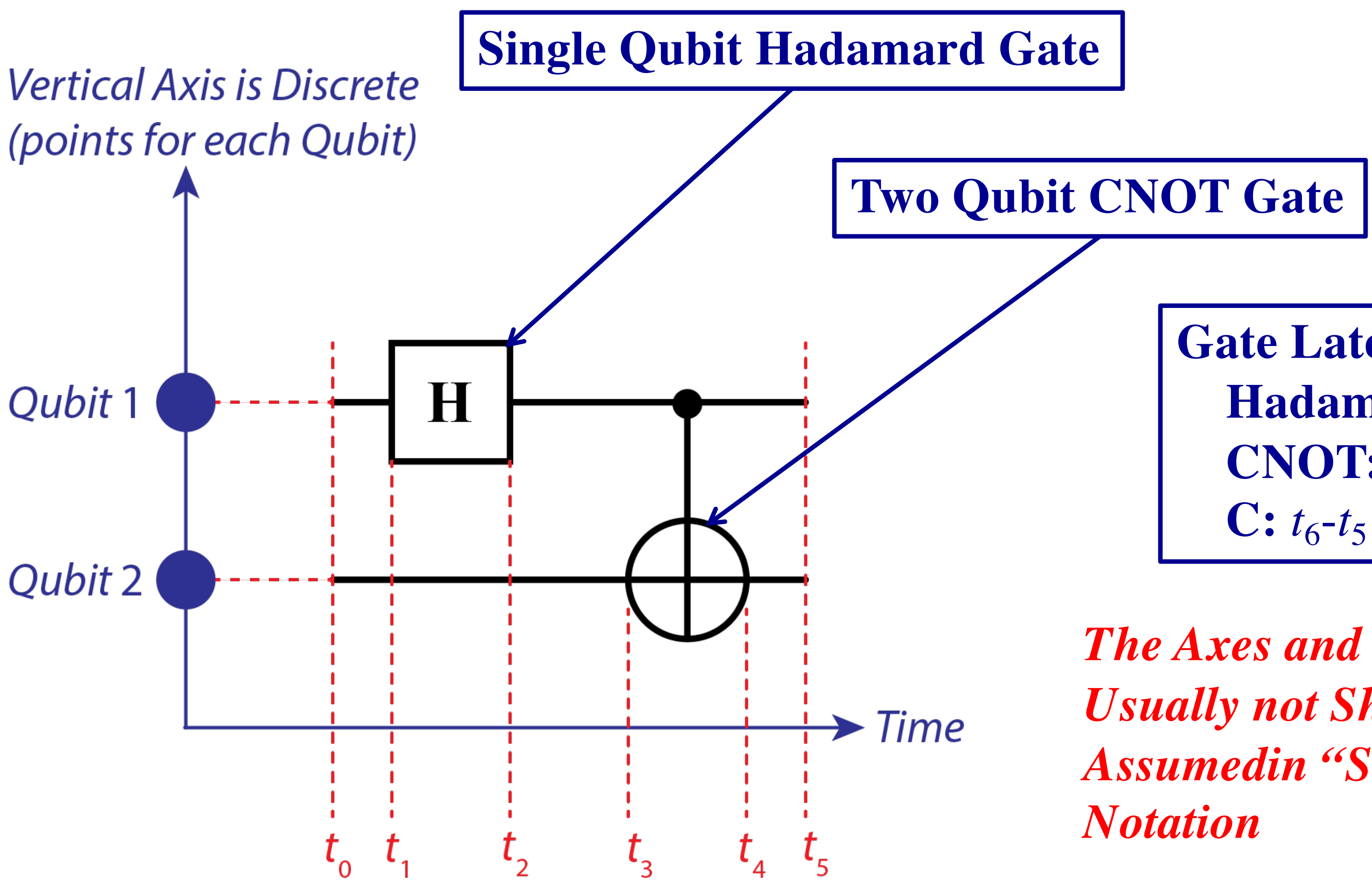
Graphical Representation

- Horizontal Lines Represent a Qubit
- Symbols that Vertically Span one or more Qubits are Quantum Gates, Operators, or QC Instructions



Graphical Representation Example

- This is Well-Known "Bell State Generator"



Gate Latencies:
Hadamard: t_2-t_1
CNOT: t_4-t_3
C: t_6-t_5

The Axes and Times are Usually not Shown, but Assumedin "Standard" Notation

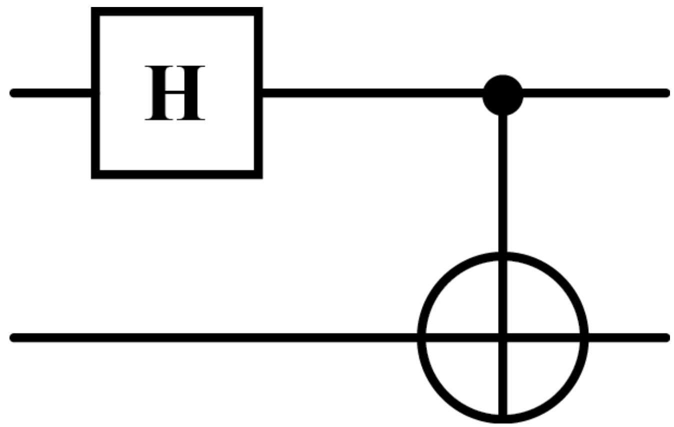
Start Time: t_0=0

End Time: t_5

Overall Delay: t_5-t_0=t_5

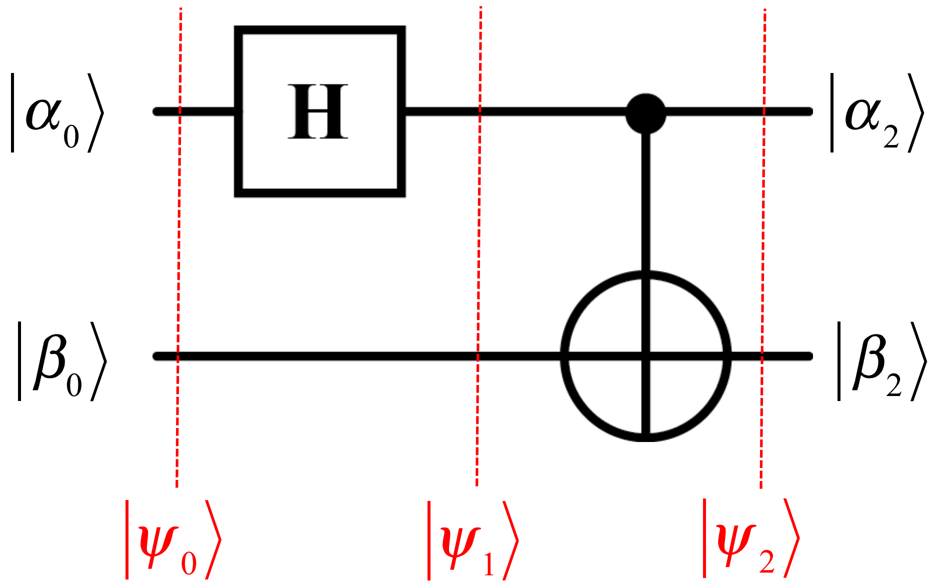
Graphical Representation Example (cont.)

- This is Well-Known "Bell State Generator"



- How is such a Circuit analyzed?
  - Qubit Values Represented as 2-dim. Column Vectors
  - Operators Represented as Linear Transformation Matrices
- Combining Qubits Accomplished with Outer Product Multiplication
- Combining Operators Accomplished with Direct Matrix Products
- Generally use Bra-Ket Notation for Conciseness

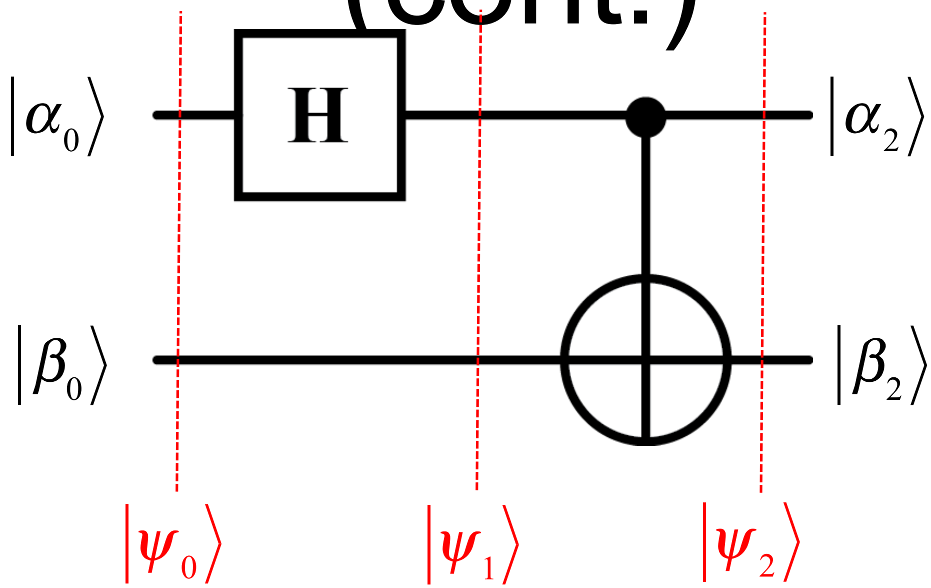
Bell State Generator Analysis



- I Prefer to Denote the Quantum State (Wave Function) at Discrete Points in Time with Dashed Lines
- STEP 1: Initialize Qubits at Time Zero to a Particular Value
  - Typically a "Ground" Basis State,

$$|\psi_0\rangle = |\alpha_0\rangle|\beta_0\rangle = |\alpha_0\rangle \otimes |\beta_0\rangle = |\alpha_0\beta_0\rangle = |00\rangle$$

Bell State Generator Analysis (cont.)



- STEP 2: Calculate Evolved Quantum States at Each Denoted Time Instant

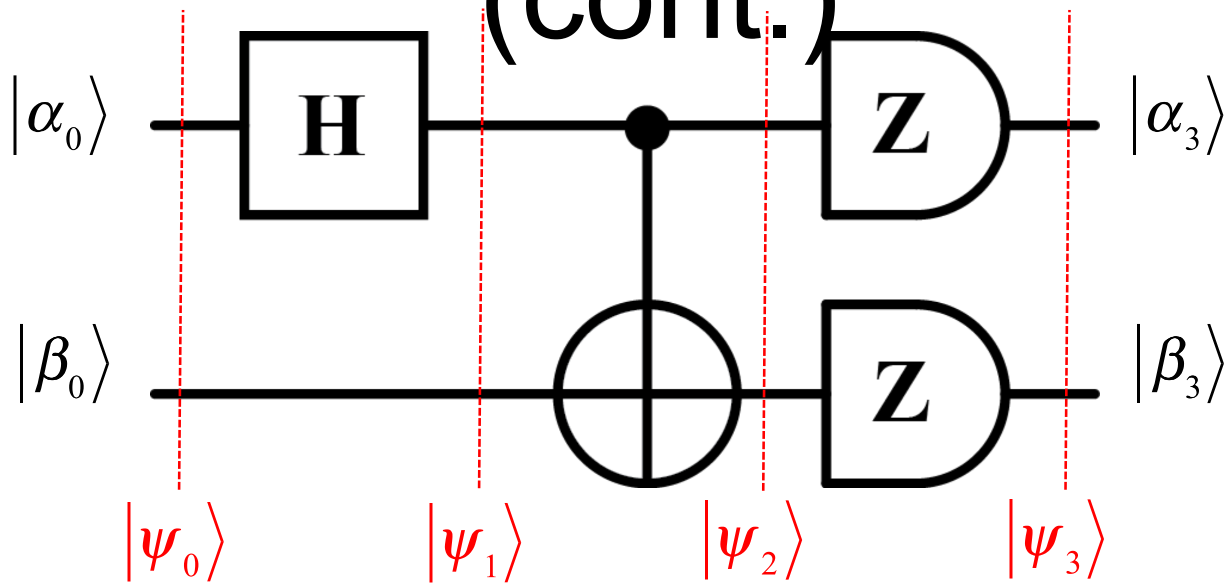
$$|\psi_2\rangle = \mathbf{C_N}|\psi_1\rangle \qquad |\psi_1\rangle = (\mathbf{H}\otimes\mathbf{I})|\psi_0\rangle \qquad |\psi_2\rangle = \mathbf{C_N}|\psi_1\rangle = \mathbf{C_N}(\mathbf{H}\otimes\mathbf{I})|\psi_0\rangle$$

- Overall Circuit/Program Transfer Matrix, **U**, is:

$$\mathbf{U} = \mathbf{C_N}(\mathbf{H}\otimes\mathbf{I})$$

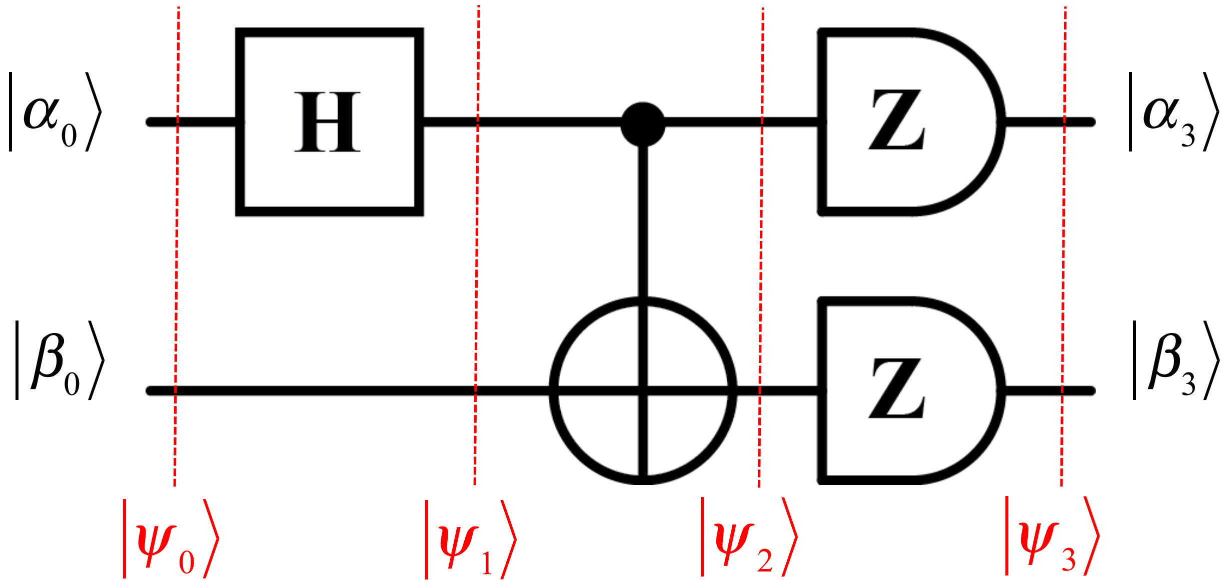
$$|\psi_2\rangle = \mathbf{U}|\psi_0\rangle$$

Bell State Generator Analysis (cont.)



- STEP 3: Measure (Observe) the Overall Evolved Quantum State (using $\mathbf{Z}$ Detectors)
  - For Photonic Circuit, use Detectors at $|\psi_2\rangle$
- For this Example, $|\psi_2\rangle$ is an Entangled & Superimposed State. Measuring $|\psi_2\rangle$ is an Evolution Yielding $|\psi_3\rangle$
  - Probability of Measuring $|\psi_3\rangle = |00\rangle$ is 0.5
  - Probability of Measuring $|\psi_3\rangle = |11\rangle$ is 0.5

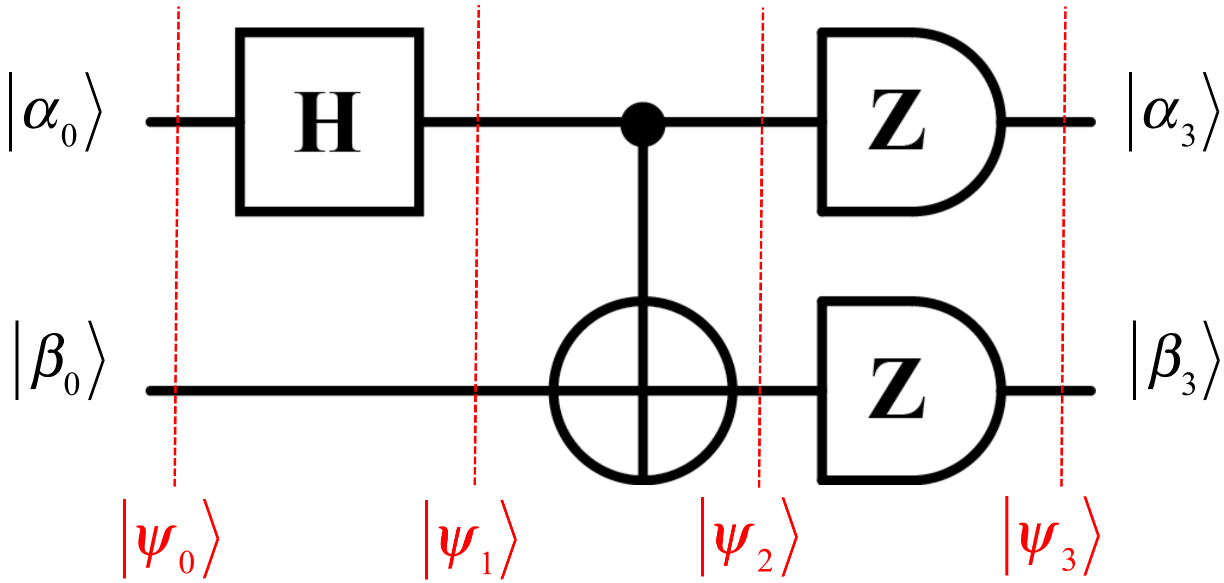
Numerical Example



$$|\psi_0\rangle = |\alpha_0\rangle|\beta_0\rangle = |\alpha_0\rangle \otimes |\beta_0\rangle = |\alpha_0\beta_0\rangle = |00\rangle$$

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

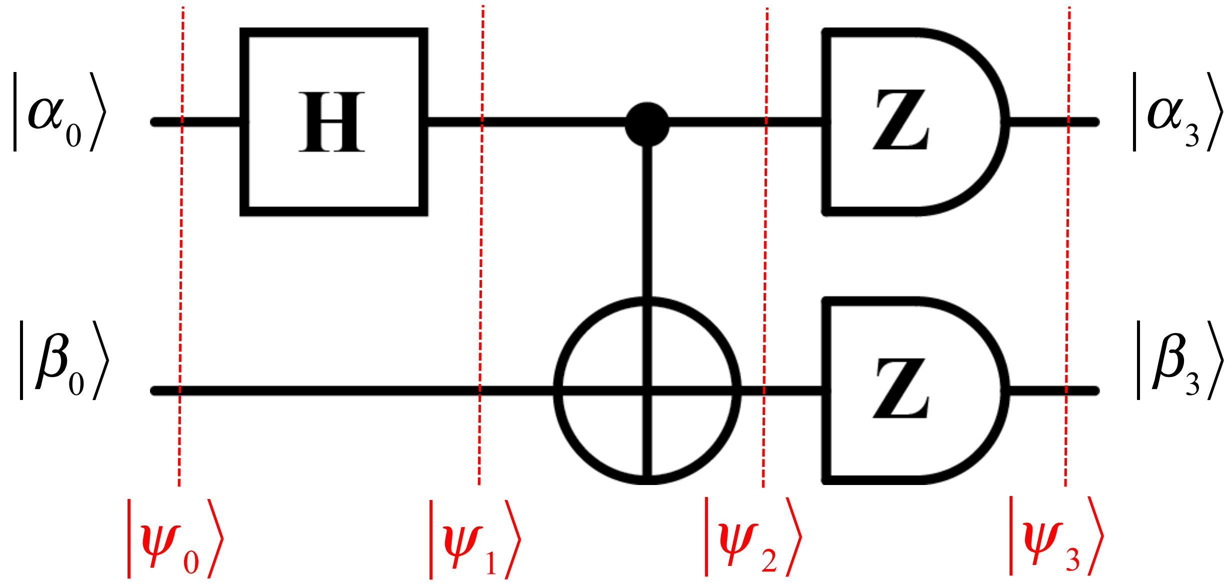
Numerical Example (cont.)



$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

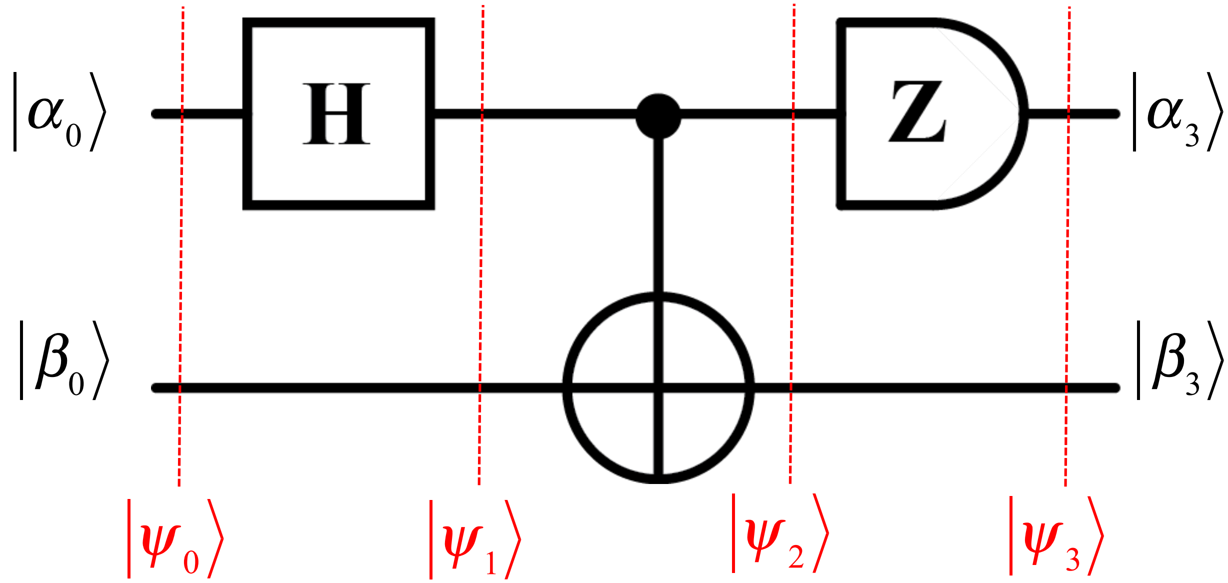
Numerical Example (cont.)



$$|\psi_2\rangle = \mathbf{U}|\psi_0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

- This is an Entangled Bell State since Impossible to Have $|\psi_3\rangle = |01\rangle$ or $|\psi_3\rangle = |10\rangle$
- Need Only Measure One Qubit to “Know” the Other

Numerical Example (cont.)



- Measurement with Respect to Computational Basis (**Z**-basis) Causes Probabilistic Collapse to One of the Eigenstates (i.e., eignevectors of Pauli-**Z**)
- If $|\alpha_3\rangle = |0\rangle$, Then $|\beta_3\rangle = |0\rangle$ Also
- If $|\alpha_3\rangle = |1\rangle$, Then $|\beta_3\rangle = |1\rangle$ Also