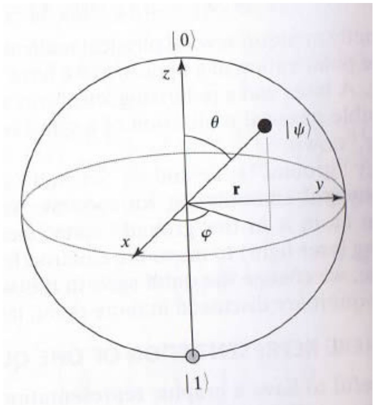
Qubits



A viewpoint of the Qubit

1

Summary so Far

- New Model Developed Since Classical Physics Could Not Explain Wave/Particle Duality
- Heisenberg/Schrödinger Developed Quantum Theory
- von Neumann Developed Mathematical Model known as Hilbert Space
- Quantum State Vector (the wave function) Represents Superposition of States

2

Summary so Far

- Quantum States Evolve over Time and Evolution Modeled as Hermitian Operators
- Observable is Attribute of Physical System that is (in principle) Measurable
- Measurement of Observable Associated with a Hermitian Operator
- Outcome of Measurement is Eigenvector of Hermitian Operator
- Hermitian eigenvalues predict which eigenvector will be measured

3

Quantum Bit

- Elementary Quantum Object used to Store Information
- For now, We view Qubit as a Mathematical Abstraction
- Qubit is a Vector in 2-D Complex Vector Space
- State of a Qubit a Superposition of Pair of Orthonormal Basis Vectors

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

4

Qubits versus Bits

Classical Bit

$$b = a_0 0 + a_1 1$$

Quantum Bit

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

- Classical Bit:
  - Only two Possible States: $(a_0=0, a_1=1)$ or $(a_0=1, a_1=0)$
- Quantum Bit:
  - Many Possible States:

$$\alpha_0, \alpha_1 \in \mathbb{C} \qquad |\alpha_0|^2 + |\alpha_1|^2 = 1$$

- Measurement of Classical Bit Yields State with Probability of 1
- Observation/Measurement of Qubit Yields:

$$|0\rangle \text{ with probability } |\alpha_0|^2 \qquad |1\rangle \text{ with probability } |\alpha_1|^2$$

5

Qubit

- Vector Length (Norm) Must be 1 for this Probability Relation to hold:

$$|\alpha_0|^2 + |\alpha_1|^2 = \alpha_0^* \alpha_0 + \alpha_1^* \alpha_1 = 1$$

- EXAMPLE:

$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

$$\text{Prob}[|0\rangle \text{ measured}] = (1/2)^2 = 25\%$$

$$\text{Prob}[|1\rangle \text{ measured}] = (\sqrt{3}/2)^2 = 75\%$$

- Superposition and Effect of Measurement Force Qubit to Lose Superposition and Collapse into an Observable Operator Eigenvector

6

Orthonormal Basis

- Qubit may be Expressed as Superposition of any Two Orthonormal Basis Vectors
- Consider:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

- In this Case:

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle = \alpha_0 \frac{|+\rangle + |-\rangle}{\sqrt{2}} + \alpha_1 \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

$$|\psi\rangle = ?\,|+\rangle + ?\,|-\rangle$$

7

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|+\rangle - |-\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} - \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle - |0\rangle + |1\rangle\right] = \frac{2}{\sqrt{2}}|1\rangle$$

$$|1\rangle = \frac{\sqrt{2}}{2}(|+\rangle - |-\rangle) = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

$$|+\rangle + |-\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} + \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle + |0\rangle - |1\rangle\right] = \frac{2}{\sqrt{2}}|0\rangle$$

$$|0\rangle = \frac{\sqrt{2}}{2}(|+\rangle + |-\rangle) = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

8

Orthonormal Basis

- Qubit may be Expressed as Superposition of any Two Orthonormal Basis Vectors
- Consider:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

- In this Case:

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle = \alpha_0 \frac{|+\rangle + |-\rangle}{\sqrt{2}} + \alpha_1 \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

$$|\psi\rangle = \frac{\alpha_0 + \alpha_1}{\sqrt{2}} |+\rangle + \frac{\alpha_0 - \alpha_1}{\sqrt{2}} |-\rangle$$

9

Single Qubit Transformations

- Single bit Transformations by Means of Operators
- Pauli Matrices Represent an Observable Describing the Spin of a Fermion in 3-D
- Denoted as:

$$\sigma_1 \text{ or } \mathbf{X} \qquad \sigma_2 \text{ or } \mathbf{Y} \qquad \sigma_3 \text{ or } \mathbf{Z}$$

- Often the Following Notation is Used:

$$\sigma_0 \text{ or } \mathbf{I}$$

10

Derivation of σ_0

- This Operator Performs an Identity Transformation of the Basis Vectors:

$$|0\rangle \mapsto |0\rangle \qquad |1\rangle \mapsto |1\rangle$$

- Computed as:

$$\sigma_0 = \mathbf{I} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$\sigma_0 = \mathbf{I} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\sigma_0 = \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

11

Derivation of σ_X

- This Operator "Flips" or "Negates" a Qubit:

$$|0\rangle \mapsto |1\rangle \qquad |1\rangle \mapsto |0\rangle$$

- Computed as:

$$\sigma_1 = \mathbf{X} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$\sigma_1 = \mathbf{X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\sigma_1 = \mathbf{X} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

12

Derivation of σ_Y

- This Operator Multiplies a Qubit by i (shifts the phase by 90 degrees) then "Flips" or "Negates" it:

$$|0\rangle \mapsto i\,|1\rangle \qquad |1\rangle \mapsto -i\,|0\rangle$$

- Computed as: $\sigma_2 = \mathbf{Y} = -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|$

$$\sigma_2 = \mathbf{Y} = -i\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0 & 1\end{bmatrix} + i\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1 & 0\end{bmatrix}$$

$$\sigma_2 = \mathbf{Y} = -i\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + i\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$$

13

Derivation of σ_Z

- This Operator is an Identity with a Negation Operation (180 degree phase shift):

$$|0\rangle \mapsto |0\rangle \qquad |1\rangle \mapsto -|1\rangle$$

- Computed as:

$$\sigma_3 = \mathbf{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\sigma_3 = \mathbf{Z} = \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1 & 0\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0 & 1\end{bmatrix}$$

$$\sigma_3 = \mathbf{Z} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

14

Pauli Operator Examples

- Assume the Following:

$$|\varphi\rangle = \sigma_i \, |\psi\rangle = \sigma_i[\alpha_0 \, |0\rangle + \alpha_1 \, |1\rangle]$$

$$|\varphi\rangle = \sigma_0 \, |\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

$$|\varphi\rangle = \sigma_1 \, |\psi\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_0 \end{bmatrix}$$

$$|\varphi\rangle = \sigma_2 \, |\psi\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = i \begin{bmatrix} -\alpha_1 \\ \alpha_0 \end{bmatrix}$$

$$|\varphi\rangle = \sigma_3 \, |\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ -\alpha_1 \end{bmatrix}$$

15

Hadamard Operator

- This Operator is Commonly used to Maximize Superposition of a Qubit in a Basis State

$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Example:

$$|\psi\rangle = 0 \, |0\rangle + 1 \, |1\rangle = |1\rangle$$

$$\mathbf{H} \, |\psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = (1/\sqrt{2}) \, |0\rangle - (1/\sqrt{2}) \, |1\rangle$$

$$\text{Prob}[|0\rangle \text{ measured}] = (1/\sqrt{2})^2 = 50\%$$

$$\text{Prob}[|1\rangle \text{ measured}] = (1/\sqrt{2})^2 = 50\%$$

16

Operator Commutativity

- In General Matrix Multiplication Does not Commute

$$\mathbf{AB} \neq \mathbf{BA}$$

- Test for Operator Commutativity

$$[\mathbf{A},\mathbf{B}] = \mathbf{AB} - \mathbf{BA} \qquad [\mathbf{A},\mathbf{B}] = \begin{cases} \mathbf{0}, & \mathbf{A},\mathbf{B} \text{ commute} \\ & \text{do not commute} \end{cases}$$

- Commutativity Test for Pauli Matrices:

$$[\sigma_1,\sigma_2] = 2i\mathbf{Z} \qquad [\sigma_2,\sigma_3] = 2i\mathbf{X} \qquad [\sigma_3,\sigma_1] = 2i\mathbf{Y}$$

Does $\sigma_i\sigma_i$ Commute?

17

Quantum Interference

Lecture From Prof. David Deutsch (42 min)

18

Geometrical Interpretation

- Bloch Sphere is Geometric Interpretation of Qubit State and Single Qubit Operators
- Express State of Qubit using 3 Real Values Interpreted as Angles

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

$$\alpha_0 = e^{i\gamma} \cos\frac{\theta}{2} \qquad \alpha_1 = e^{i\gamma} e^{i\varphi} \sin\frac{\theta}{2}$$

$$|\psi\rangle = e^{i\gamma} \left[\cos\frac{\theta}{2} |0\rangle + e^{i\varphi} \sin\frac{\theta}{2} |1\rangle \right]$$

19

Geometrical Interpretation

- Recall that Phase Factors are not Observable
- Can be Ignored for our Calculations
- Verify Norm of State Vector is Unity

$$|\alpha_0|^2 + |\alpha_1|^2 = 1$$

$$\alpha_0 = e^{i\gamma} \cos\frac{\theta}{2} \qquad \alpha_1 = e^{i\gamma} e^{i\varphi} \sin\frac{\theta}{2}$$

20

Geometrical Interpretation

$$|\alpha_0|^2 + |\alpha_1|^2 = 1$$

$$\left|e^{i\gamma}\cos\frac{\theta}{2}\right|^2 + \left|e^{i\gamma}e^{i\varphi}\sin\frac{\theta}{2}\right|^2 = |e^{i\gamma}|^2\cos^2\frac{\theta}{2} + |e^{i\gamma}|^2|e^{i\varphi}|^2\sin^2\frac{\theta}{2}$$

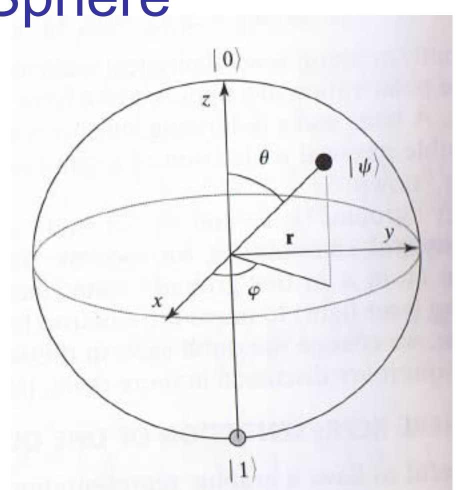
$$= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$$

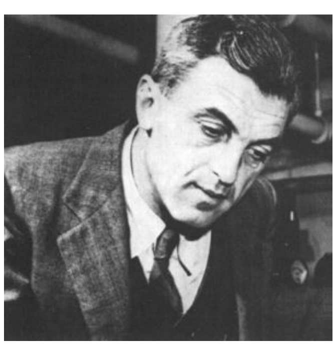
Note that: $|e^{i\gamma}|^2 = |e^{i\varphi}|^2 = 1$

21

Bloch Sphere

- Qubit State is Vector from Origin to Point on Unit Sphere
- Position of Point Defined by 2 Real-valued Angles
- Named after Physicist Felix Bloch





First Director of CERN
Studied with Heisenberg, Pauli, Bohr, Fermi
1961: Max Stein Chair Stanford

22

Bloch Sphere

EXAMPLE:

$$|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$$

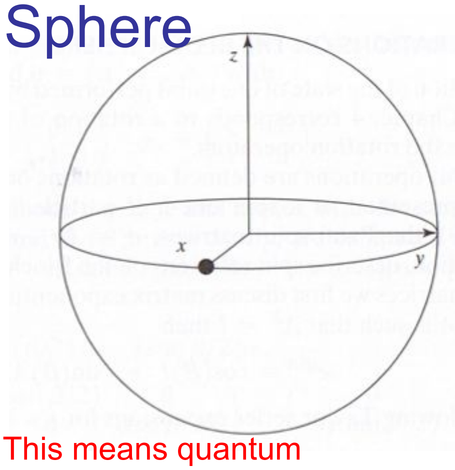
$$\alpha_0 = \alpha_1 = 1/\sqrt{2}$$

$$\alpha_0 = \cos\frac{\theta}{2} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{\theta}{2} = 45^{\circ} \Rightarrow \theta = 90^{\circ}$$

$$\alpha_1 = e^{i\varphi}\sin\frac{\theta}{2} = \frac{1}{\sqrt{2}}$$

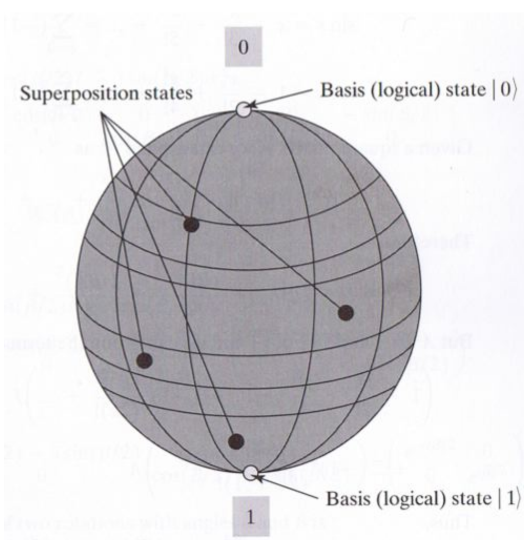
$$\Rightarrow e^{i\varphi} = 1 \Rightarrow \varphi = 0^{\circ}$$



This means quantum state lies along the positive y-axis in the Bloch sphere which is an even superposition of the two eignekets or the eigenbasis

23

Bloch Sphere



24

Rotations on Bloch Sphere

- Single Qubit state Transformation Corresponds to Rotation over Bloch Sphere
- When Qubit is Represented by Fermion, Pauli Matrices Describe Rotations
- Need Another Mathematical Tool:
  - Matrix Exponentiation

25

Matrix Exponentiation

- Assume $\mathbf{A}$ is a Matrix such that $\mathbf{A}^2 = \mathbf{I}$ and β is a Real Number, then:

$$e^{i\beta\mathbf{A}} = \cos(\beta)\mathbf{I} + i\sin(\beta)\mathbf{A}$$

- Recall Taylor Expansion Series for: $x \in \sim$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

26

Taylor Expansion of Matrix

- Assume $\mathbf{A}$ is a Square Matrix, then:

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \cdots + \frac{\mathbf{A}^k}{k!} + \cdots$$

$$e^{i\beta\mathbf{A}} = \mathbf{I} + (i\beta\mathbf{A}) + \frac{(i\beta\mathbf{A})^2}{2!} + \frac{(i\beta\mathbf{A})^3}{3!} + \cdots + \frac{(i\beta\mathbf{A})^k}{k!} + \cdots$$

- Note that: $\mathbf{A}^2 = \mathbf{I} \qquad i = \sqrt{-1}$
- We Regroup the Terms as Shown on the Following:

27

Taylor Expansion of Matrix

- Regrouping:

$$e^{i\beta\mathbf{A}} = \mathbf{I} + (i\beta\mathbf{A}) + \frac{(i\beta\mathbf{A})^2}{2!} + \frac{(i\beta\mathbf{A})^3}{3!} + \cdots + \frac{(i\beta\mathbf{A})^k}{k!} + \cdots$$

$$e^{i\beta\mathbf{A}} = \left(1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} + \cdots + (-1)^k \frac{(\beta)^{2k}}{(2k)!} + \cdots\right)\mathbf{I}$$

$$+ i\left(\beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} + \cdots + (-1)^k \frac{(\beta)^{2k+1}}{(2k+1)!} + \cdots\right)\mathbf{A}$$

$$e^{i\beta\mathbf{A}} = \cos(\beta)\mathbf{I} + i\sin(\beta)\mathbf{A}$$

28

Pauli Rotations

- Consider a Finite Rotation through an Angle β about a given Vector $\mathbf{n}$ on the Bloch Sphere:

$$\mathcal{R}_{\mathbf{n}}(\beta) = \exp(-i(\beta/2)\mathbf{n} \bullet \sigma) = \cos(\beta/2)\mathbf{I} - i\sin(\beta/2)\mathbf{n} \bullet \sigma$$

- In this Operator, note the following:

$\mathbf{n} \bullet \sigma$ yields a matrix and $\mathbf{n} = (n_x, n_y, n_z), \sigma = (\sigma_{\mathbf{X}}, \sigma_{\mathbf{Y}}, \sigma_{\mathbf{Z}})^{\mathrm{T}}$

$$\sigma_{\mathbf{X}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_{\mathbf{Y}} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_{\mathbf{Z}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

29

Pauli Rotations

- Rotation Operators about x, y, and z Axes through Angle β are Denoted as:

$$\mathcal{R}_x(\beta), \mathcal{R}_y(\beta), \mathcal{R}_z(\beta)$$

- Observing that:

$$\mathbf{n} \bullet \sigma = [(1,0,0) \bullet (\sigma_{\mathbf{X}}, 0, 0)^{\mathrm{T}}], [(0,1,0) \bullet (0, \sigma_{\mathbf{Y}}, 0)^{\mathrm{T}}], [(0,0,1) \bullet (0,0,\sigma_{\mathbf{Z}})^{\mathrm{T}}]$$

- Using this Observation and the Previous Result: $\mathcal{R}_x(\beta) = cos(\beta/2)\mathbf{I} - isin(\beta/2)\boldsymbol{\sigma}_x$

$$\mathcal{R}_x(\beta) = \begin{bmatrix} \cos(\beta/2) & 0 \\ 0 & \cos(\beta/2) \end{bmatrix} + \begin{bmatrix} 0 & -i\sin(\beta/2) \\ -i\sin(\beta/2) & 0 \end{bmatrix}$$

30

Pauli Rotations (cont)

$$\mathcal{R}_x(\beta) = \begin{bmatrix} \cos(\beta/2) & -i\sin(\beta/2) \\ -i\sin(\beta/2) & \cos(\beta/2) \end{bmatrix}$$

- Similar Derivations Yield:

$$\mathcal{R}_y(\beta) = \begin{bmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{bmatrix}$$

$$\mathcal{R}_z(\beta) = \begin{bmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{-i\beta/2} \end{bmatrix}$$

31

Rotation Operator Properties

- Angle Addition Property:

$$\mathcal{R}_x(\beta_1)\mathcal{R}_x(\beta_2) = \mathcal{R}_x(\beta_1+\beta_2)$$
$$\mathcal{R}_y(\beta_1)\mathcal{R}_y(\beta_2) = \mathcal{R}_y(\beta_1+\beta_2)$$
$$\mathcal{R}_z(\beta_1)\mathcal{R}_z(\beta_2) = \mathcal{R}_z(\beta_1+\beta_2)$$

- Recall from Trigonometry:

$$\sin(\beta_1 \pm \beta_2) = \sin(\beta_1)\cos(\beta_2) \pm \cos(\beta_1)\sin(\beta_2)$$
$$\cos(\beta_1 \pm \beta_2) = \cos(\beta_1)\cos(\beta_2) \mp \sin(\beta_1)\sin(\beta_2)$$

$$\mathcal{R}_y(\beta_1)\mathcal{R}_y(\beta_2) = \begin{bmatrix} \cos(\beta_1/2) & -\sin(\beta_1/2) \\ \sin(\beta_1/2) & \cos(\beta_1/2) \end{bmatrix} \begin{bmatrix} \cos(\beta_2/2) & -\sin(\beta_2/2) \\ \sin(\beta_2/2) & \cos(\beta_2/2) \end{bmatrix}$$

$$\mathcal{R}_y(\beta_1)\mathcal{R}_y(\beta_2) = \begin{bmatrix} \cos[(\beta_1+\beta_2)/2] & -\sin[(\beta_1+\beta_2)/2] \\ \sin[(\beta_1+\beta_2)/2] & \cos[(\beta_1+\beta_2)/2] \end{bmatrix} = \mathcal{R}_y(\beta_1+\beta_2)$$

32

Qubit Measurement

- In General Qubits are in Superposition State:

$$|\psi\rangle = \alpha_0\,|0\rangle + \alpha_1\,|1\rangle$$

- Measurement Characterized by set of Linear Operators that are Modeled as Hermitian Matrices

$$\{\mathcal{M}_k\}$$

- Probability of Outcome with Index k as Result of Measurement is:

$$p(k) = \langle\psi\,|\mathcal{M}_k\mathcal{M}\ |\psi\rangle$$

33

Qubit Measurement

- All Possible Measurements:

$$\sum_k p(k) = \sum_k \langle\psi\,|\mathcal{M}_k\mathcal{M}_k\,|\,\psi\rangle = 1$$

- Classical Probability is Used when there are “missing details”
- Appears that this is Not True in QM, it Occurs Naturally in Models as we Understand Them

34

Measurement on Bloch Sphere

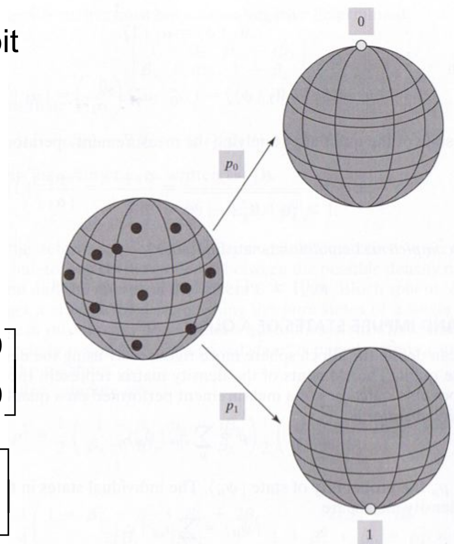
- Measurement Causes Qubit to Change State:

$$|\psi\rangle \mapsto |\varphi\rangle = \frac{\mathcal{M}_k |\psi\rangle}{\sqrt{\langle\psi|\mathcal{M}_k^\dagger \mathcal{M}_k|\psi\rangle}}$$

- Two Possible Outcomes
- Measurement Operators:

$$\mathcal{M}_0 = |0\rangle\langle 0| = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

$$\mathcal{M}_1 = |1\rangle\langle 1| = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$



Possible states of one qubit before the measurement

The state of the qubit after the measurement

35

Measurement on Bloch Sphere

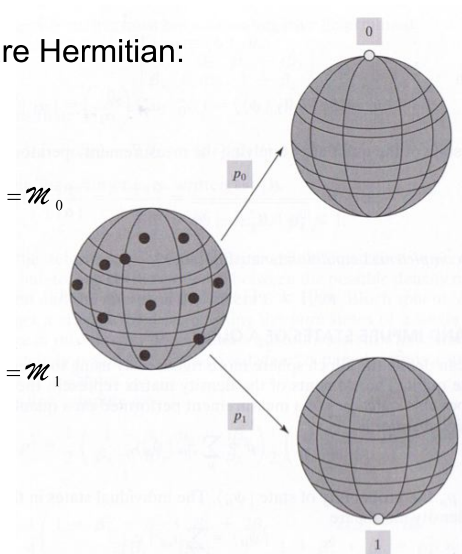
- Measurement Operators are Hermitian:

$$\mathcal{M}_0^\dagger = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} = \mathcal{M}_0$$

$$\mathcal{M}_0^\dagger \mathcal{M}_0 = \mathcal{M}_0^2 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} = \mathcal{M}_0$$

$$\mathcal{M}_1^\dagger = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \mathcal{M}_1$$

$$\mathcal{M}_1^\dagger \mathcal{M}_1 = \mathcal{M}_1^2 = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \mathcal{M}_1$$



Possible states of one qubit before the measurement

The state of the qubit after the measurement

36

Measurement

- Probability of Outcome Corresponding to ket-zero:

$$p_0 = \langle \psi | \mathcal{M}_0^\dagger \mathcal{M}_0 | \psi \rangle = \langle \psi | \mathcal{M}_0 | \psi \rangle$$

$$\mathcal{M}_0 | \psi \rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ 0 \end{bmatrix}$$

$$p_0 = \langle \psi | (\mathcal{M}_0 | \psi \rangle) = \begin{bmatrix} \alpha_0^* & \alpha_1^* \end{bmatrix} \begin{bmatrix} \alpha_0 \\ 0 \end{bmatrix} = | \alpha_0 |^2$$

- Also

$$p_1 = \langle \psi | (\mathcal{M}_1 | \psi \rangle) = \begin{bmatrix} \alpha_0^* & \alpha_1^* \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_1 \end{bmatrix} = | \alpha_1 |^2$$

37

Measurement

- Note that:

$$\mathcal{M}_0 | \psi \rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ 0 \end{bmatrix} = \alpha_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \alpha_0 | 0 \rangle$$

- State of Qubit after Measurement is:

$$| \psi \rangle \mapsto | \varphi_0 \rangle = \frac{\mathcal{M}_0 | \psi \rangle}{\sqrt{\langle \psi | \mathcal{M}_0^\dagger \mathcal{M}_0 | \psi \rangle}} = \frac{\alpha_0 | 0 \rangle}{| \alpha_0 |} = \frac{\alpha_0}{| \alpha_0 |} | 0 \rangle$$

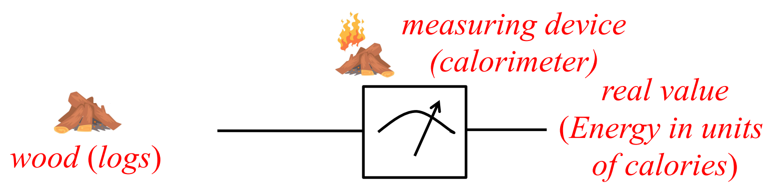
- Likewise:

$$| \psi \rangle \mapsto | \varphi_1 \rangle = \frac{\mathcal{M}_1 | \psi \rangle}{\sqrt{\langle \psi | \mathcal{M}_1^\dagger \mathcal{M}_1 | \psi \rangle}} = \frac{\alpha_1 | 1 \rangle}{| \alpha_1 |} = \frac{\alpha_1}{| \alpha_1 |} | 1 \rangle$$

38

Measurements Require:

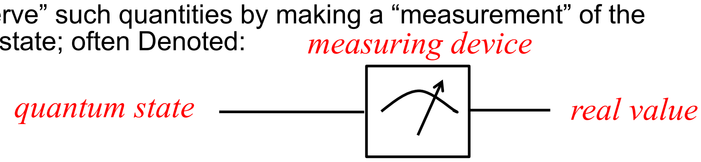
- **INPUT**: Object to be observed or measured
- **OUTPUT**: A Real-value called the "measurement outcome"
- **METHOD**: Choosing the appropriate device to perform the measurement/observation
- *Destructive measurements* affect the object (change its form) but the outcome is NOT the object, it is the real-value or measurement outcome; object transformation is a side effect
  – example: *measuring* the *heat capacity* (in units of energy) of a piece of wood requires measuring the amount of temperature increase over time as the wood is converted in form from a solid to a gas (i.e., "Burning" the wood)



39

Observables and Measurement

- Quantum observables are anything we can observe about a quantum object; a quantity like position, energy, momentum, etc.
- We "observe" such quantities by making a "measurement" of the quantum state; often Denoted:



- QM postulate states that some observables can only be probabilistically measured. This is Born's rule.
  – Such as measuring the value of a qubit. Thus the real value is a single classical bit, but measurement outcome can differ among multiple measurements.
- Quantum State measurements are *Absolute measurements* and depend upon the frame of reference (vector space basis)
- Quantum State measurements are *Destructive*; they cause the quantum state to change

40

Observables are Mathematical

- Recall that Mathematical Operators exist that can be applied to a quantum state to yield different Observables

Observable Name	Observable Symbol	Operator Symbol	Operator Operation
Position	$\underline{r}$	$\hat{\mathbf{r}}$	Multiply by $\underline{r}$
Momentum	$\mathbf{p}$	$\hat{\mathbf{p}}$	$-i\hbar\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)$
Kinetic energy	T	$\hat{T}$	$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$
Potential energy	$V(\mathbf{r})$	$\hat{V}(\mathbf{r})$	Multiply by $V(\mathbf{r})$
Total energy	E	$\hat{H}$	$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+V(\mathbf{r})$
Angular momentum	l_x	$\hat{l}_x$	$-i\hbar\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)$
	l_y	$\hat{l}_y$	$-i\hbar\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)$
	l_z	$\hat{l}_z$	$-i\hbar\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)$

Photon Location

Superconducting Solid-State Circuits

Ion Traps

Other Photon Properties (polarization, OAM; orbital angular momentum)

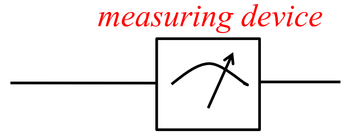
- For this reason, an Observable is ALSO a Mathematical Operator applied to a quantum state

41

Notation can be Confusing

- *Measurement*:

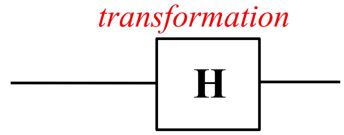
measuring device



- *Quantum State transformation* (or evolution):

$$\mathbf{H} = e^{\frac{-i}{\hbar}\hat{\mathbf{H}}(t_1-t_0)} \qquad \left|\boldsymbol{\Psi}(t_1)\right\rangle = e^{\frac{-i}{\hbar}\hat{\mathbf{H}}(t_1-t_0)}\left|\boldsymbol{\Psi}(t_0)\right\rangle$$

transformation



- Measurements also change the quantum state, but the OUTCOME is a real-value. The quantum state is destructively changed according to a probability distribution.
- If state change due to measurement is IDEAL (no energy loss to environment), we have a *Projective Measurement*

42

Transformations versus Measurements

- *Projective Measurements* cause the Quantum State change to conserve energy in a closed system
  - the destructive measurement transforms the state according to an evolution with no energy loss to environment
- *General Measurements* account for realistic energy loss to environment
  - energy loss to the environment is decoherence
  - destructive state change is NOT modeled as a time evolution in a closed system
- *Positive/Probability Operator-Valued Measures* (*POVM*) are Intermediate Measurement model that accounts for energy loss to the environment by assuming the Vector Space is of additional dimension to account for states "lost" to the environment, $\mathbf{H}_{n+m}$, ($\mathbf{H}_{2+m}$ for qubits)
  - quantum state is n-dimensional, environment adds m dimensions to account for case where destructive measurement causes energy to transfer outside the closed quantum system
  - Ancilla state in m-dimensions is added to the system to account for potential energy transfer to the environment (decoherence) during a measurement

We Focus on Projective Measurements First

43

Example: Measuring Probability Amplitudes

- We ***cannot*** directly observe/measure the probability amplitudes
  - doing so would mean a single qubit can store more information than a single classical bit from an Information Theory point of view
  - this is a consequence of a QM postulate that is known as Born's rule, probabilistic observations
  - if we could conduct a large number of measurements of the same qubit, we could obtain an estimate of the probabilities, the square of the magnitude of the probability amplitudes; however, since measurements are *generally destructive*, this is not possible
- Superposition allows a qubit to "represent" both zero and one, but a single measurement causes the qubit to "collapse" into a single probabilistic outcome (0 or 1).
  - State collapse is a characterization of the Destructive nature of the measurement

44

Measuring the Quantum State

https://www.youtube.com/watch?v=SMbh0GgCN7I (11:29)

45

Born's Rule (QM Postulate)

- Consider a Qubit in Terms of the Computational Basis:

$$|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

- Probability of Observing a Qubit Basis State is the Square of Magnitude of Probability Amplitude
  - This is a ***Postulate*** of QM (Born's Rule)

$$\text{Prob}\left[|\Psi\rangle \rightarrow |0\rangle\right] = |\alpha|^2 \qquad \text{Prob}\left[|\Psi\rangle \rightarrow |1\rangle\right] = |\beta|^2$$

- Consider the inner product of a Qubit with itself:

$$\langle\Psi|\Psi\rangle = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$ by the Born's rule Postulate:

$$= \alpha^*\alpha + \beta^*\beta = |\alpha|^2 + |\beta|^2$$

46

Born's Rule (QM Postulate)

- Consider Qubit in Terms of the Computational Basis:

$$|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

- Probability of Observing a Qubit Basis State is the Square of Magnitude of Probability Amplitude
  - This is a ***Postulate*** of QM (Born's Rule)

$$\mathrm{Prob}\Big[|\Psi\rangle \to |0\rangle\Big] = |\alpha|^2 \qquad \mathrm{Prob}\Big[|\Psi\rangle \to |1\rangle\Big] = |\beta|^2$$

- Consider the inner product of a Qubit with itself:

$$\langle\Psi|\Psi\rangle = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$ by the Born's rule Postulate:

$$= \alpha^*\alpha + \beta^*\beta = |\alpha|^2 + |\beta|^2$$
$$= \mathrm{Prob}\Big[|\Psi\rangle \to |0\rangle\Big] + \mathrm{Prob}\Big[|\Psi\rangle \to |1\rangle\Big]$$
$$= 1$$

47

Projecting a Probability Amplitude

- Consider Qubit in Terms of the Computational Basis:

$$|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

- If we wished to *mathematically project* the qubit onto the $|0\rangle$, we could formulate a Projector (projection matrix), $\mathbf{P}_0$, and we could compute:

$$\mathbf{P}_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$|\Psi_0\rangle = \mathbf{P}_0|\Psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \alpha|0\rangle$$

- Note that this is **NOT** a valid quantum operator as $\mathbf{P}_0$ is **NOT** unitary and thus **NOT** a solution of the time-dependent Schrödinger Equation, but it is Mathematically a valid Projection Matrix:
- Also note that $\mathbf{P}_0$ is a Hermitian Projection Matrix.

48

Born's Rule Again

- Consider Qubit in Terms of the Computational Basis:

$$|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

- Recall that the Probability of Observing the $|0\rangle$ basis state is the Square of the Magnitude of the $|0\rangle$ Probability Amplitude (QM postulate, cannot derive), thus:

$$\text{Prob}\left[|\Psi\rangle \rightarrow |0\rangle\right] = |\alpha|^2$$

- From Previous Slide: $|\Psi_0\rangle = \mathbf{P}_0|\Psi\rangle = \alpha|0\rangle$
- We can express this in terms of the norm of the qubit projection to $|0\rangle$ as:

$$|\alpha|^2 = \|\,|\Psi_0\rangle\,\|^2 = \langle\Psi_0|\Psi_0\rangle = \begin{bmatrix} \alpha^* & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \alpha^*\alpha$$

- Or, likewise in terms of the projection operator matrix $\mathbf{P}_0$ as:

$$|\alpha|^2 = \left\|\mathbf{P}_0|\Psi\rangle\right\|^2 = \left(\mathbf{P}_0|\Psi\rangle\right)^\dagger\left(\mathbf{P}_0|\Psi\rangle\right) = \langle\Psi|\mathbf{P}_0^\dagger\mathbf{P}_0|\Psi\rangle$$

49

Hermitian Projectors

$$|\alpha|^2 = \left\|\mathbf{P}_0|\Psi\rangle\right\|^2 = \left(\mathbf{P}_0|\Psi\rangle\right)^\dagger\left(\mathbf{P}_0|\Psi\rangle\right) = \langle\Psi|\mathbf{P}_0^\dagger\mathbf{P}_0|\Psi\rangle$$

- Recall from our discussion of Projector Operators that a projection operator is an "orthogonal projection matrix" when Hermitian idempotence holds:

$$\mathbf{P}^2 = \mathbf{P} = \mathbf{P}^\dagger$$

- Given that:

$$\mathbf{P}_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{P}_0^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{P}_0 = \mathbf{P}_0^\dagger$$

- We observe that $\mathbf{P}_0$ *is Hermitian* and that *idempotence holds*, thus we can rewrite:

$$|\alpha|^2 = \langle\Psi|\mathbf{P}_0^\dagger\mathbf{P}_0|\Psi\rangle = \langle\Psi|\mathbf{P}_0|\Psi\rangle$$

- Using this result and combining with Born's Rule postulate:

$$\text{Prob}\left[|\Psi\rangle \rightarrow |0\rangle\right] = |\alpha|^2 = \langle\Psi|\mathbf{P}_0|\Psi\rangle$$

50

Complete Sets of Projectors

- Considering all Possible Projectors, we can a Complete set of projection operators.
- Completeness means that all of the Projector Operators correspond to all Possible Measurement Outcomes.
- In the case of projecting a single qubit to the *computational basis*, all possible outcomes Correspond to either measuring a $|0\rangle$ or a $|1\rangle$ with the Projector set consisting of $\{\mathbf{P}_0, \mathbf{P}_1\}$:

$$\mathbf{P}_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{P}_1 = |1\rangle\langle 1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\{\mathbf{P}_0, \mathbf{P}_1\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$$

- The Completeness requirement can also be stated as:

$$\sum_{i=1}^{n} \mathbf{P}_i = \mathbf{I}$$

51

Projector Basis Set

- Ultimately, the Complete Set of Projectors depends on a Complete Basis set that Spans the n-dimensional Vector Space that contains the Quantum State
  - for a single qubit, $n=2$, since the qubit is a quantum state in $\mathbf{H}_2$

$$\{|e_0\rangle, |e_1\rangle, \cdots, |e_{n-1}\rangle\} \Rightarrow \{\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{n-1}\} = \{|e_0\rangle\langle e_0|, |e_1\rangle\langle e_1|, \cdots, |e_{n-1}\rangle\langle e_{n-1}|\}$$

- Because there are many different Basis Sets, there are many different Complete Sets of Projectors
- EXAMPLE: For a single qubit, $n=2$, and using the Computational Basis, we have:

$$\{|e_0\rangle, |e_1\rangle\} = \{|0\rangle, |1\rangle\} \Rightarrow \{\mathbf{P}_0, \mathbf{P}_1\} = \{|0\rangle\langle 0|, |1\rangle\langle 1|\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\mathbf{P}_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{P}_1 = |1\rangle\langle 1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\{\mathbf{P}_0, \mathbf{P}_1\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} = \{|0\rangle\langle 0|, |1\rangle\langle 1|\} \qquad \sum_{i=1}^{n} \mathbf{P}_i = \mathbf{I}$$

52

Statistics of Quantum State

- Born's Rule indicates:

$$|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \qquad \mathrm{Prob}\Big[\,|\Psi\rangle \to |0\rangle\Big] = |\alpha|^2 \qquad \mathrm{Prob}\Big[\,|\Psi\rangle \to |1\rangle\Big] = |\beta|^2$$

- By Previous derivation, in terms of Projectors, we observed:

$$|\alpha|^2 = \left\|\mathbf{P}_0|\Psi\rangle\right\|^2 = \left(\mathbf{P}_0|\Psi\rangle\right)^{\dagger}\left(\mathbf{P}_0|\Psi\rangle\right) = \langle\Psi|\mathbf{P}_0^{\dagger}\mathbf{P}_0|\Psi\rangle = \langle\Psi|\mathbf{P}_0|\Psi\rangle$$

$$|\beta|^2 = \left\|\mathbf{P}_1|\Psi\rangle\right\|^2 = \left(\mathbf{P}_1|\Psi\rangle\right)^{\dagger}\left(\mathbf{P}_1|\Psi\rangle\right) = \langle\Psi|\mathbf{P}_1^{\dagger}\mathbf{P}_1|\Psi\rangle = \langle\Psi|\mathbf{P}_1|\Psi\rangle$$

$$|\alpha|^2 + |\beta|^2 = \langle\Psi|\mathbf{P}_0|\Psi\rangle + \langle\Psi|\mathbf{P}_1|\Psi\rangle = 1$$

- Therefore, Born's Rule allows a *Probability Mass Function* (*pmf*) to be derived in terms of Projectors:

$$\mathrm{Prob}\Big[\,|\Psi\rangle \to |\Psi_i\rangle\Big] = \langle\Psi|\mathbf{P}_i|\Psi\rangle$$

- From linear algebra, the pmf can also be expressed as:

$$\mathrm{Prob}\Big[\,|\Psi\rangle \to |\Psi_i\rangle\Big] = \langle\Psi|\mathbf{P}_i|\Psi\rangle = \mathrm{Trace}\Big[|\Psi\rangle\langle\Psi|\mathbf{P}_i\Big]$$

53

Expected Value of Quantum State

- We can now express the Quantum State Distribution or *Cummulative Density Function* function as a Summation over all the Projectors in a Complete Set of n Projectors (for some arbitrary ordering of Projectors) as:

$$\mathrm{Prob}\Big[|\Psi\rangle \le |\Psi_i\rangle\Big] = \sum_{i=1}^{m\le n}\langle\Psi|\mathbf{P}_i|\Psi\rangle \qquad \mathrm{Prob}\Big[|\Psi\rangle \le |\Psi_i\rangle\Big]\sum_{i=1}^{m=n}\langle\Psi|\mathbf{P}_i|\Psi\rangle = \mathbf{I}$$

- The *Expected Value* is expressed in BraKet notation as:

$$\langle\mathbf{P}_i\rangle_{\Psi} = \langle\Psi|\mathbf{P}_i|\Psi\rangle = \mathrm{Trace}\Big[|\Psi\rangle\langle\Psi|\mathbf{P}_i\Big]$$

- or simply:

$$\langle\mathbf{P}_i\rangle = \langle\Psi|\mathbf{P}_i|\Psi\rangle = \mathrm{Trace}\Big[|\Psi\rangle\langle\Psi|\mathbf{P}_i\Big]$$

54

Observables

https://www.youtube.com/watch?v=eUbbCpqWh-M (12:54)

55

Mathematical Defn of Observable

- An Observable, $\mathbf{A}$, can be defined as a matrix operator that is defined over a Complete set of projection operators.
- The Basis Set (used to create the Projectors) specifies the set of Measurement Outcomes we are attempting to Observe
- An Observable for a Projective Measurement consists of all Possible Measurement Outcomes and is thus a Basis Set of Vectors Spanning the Vector Space of the Quantum State
- In the case of Measuring a Qubit to Determine if it is in one of the Computational Basis States, we choose the Computational Basis vectors to serve as the Eigenvectors of our Observable, $\mathbf{A}$:

$$\{|e_0\rangle, |e_1\rangle\} = \{|0\rangle, |1\rangle\}$$

- These are used to form the Projectors, $\{\mathbf{P}_0, \mathbf{P}_1\}$:

$$\mathbf{P}_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{P}_1 = |1\rangle\langle 1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\{\mathbf{P}_0, \mathbf{P}_1\} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$$

- The Resulting Observable Operator is:

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{P}_i = \sum_{i=1}^{2} \lambda_i \mathbf{P}_i = (1)\mathbf{P}_0 + (1)\mathbf{P}_1 = (1)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (1)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

56

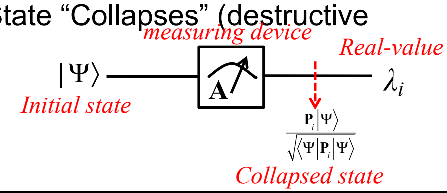
Mathematical Defn of Observable (cont.)

- An Observable, $\mathbf{A}$, can be defined as a matrix operator that is defined over a Complete set of projection operators.

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{P}_i = \sum_{i=1}^{2} \lambda_i \mathbf{P}_i = (1)\mathbf{P}_0 + (1)\mathbf{P}_1 = (1)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (1)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- The Eigenvalues are those of the Observable, $\mathbf{A}$.
- Recall the Spectral Decomposition theorem of a Hermitian Matrix: $\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{P}_i = \sum_{i=1}^{n} \lambda_i |e_i\rangle\langle e_i|$
- The Spectral Decomposition holds for the Observable since the Projectors are Hermitian, and hence the Observable $\mathbf{A}$ is Hermitian
- The Projective Measurement Outcome is the Real-valued Eigenvalue, and the Quantum State "Collapses" (destructive measurement)

$$|\Psi\rangle \rightarrow \frac{\mathbf{P}_i|\Psi\rangle}{\sqrt{\langle\Psi|\mathbf{P}_i|\Psi\rangle}}$$

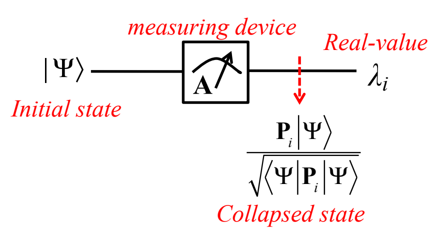


57

Measurement "Collapses" the Qubit

- Each Vector used to form the Projector Operator, $\mathbf{P}_i$, that Comprises the Observable, $\mathbf{A}$ is an Eigenvector, $|e_i\rangle$.
  - Thus, the Complete set of Projectors $\{\mathbf{P}_i | i=1,\ldots,n\}$ is formed from n Orthogonal Eigenvectors, $|e_i\rangle$, that Span a Vector Space wherein the Quantum State is an Element.
- Measurement of a Qubit results in the Qubit "collapsing" to one of the Eigenvectors of the Observable $\mathbf{A}$.
- The Destructive "Collapse" is a ***result of the Born's Rule QM Postulate and is not something that can be Derived***!

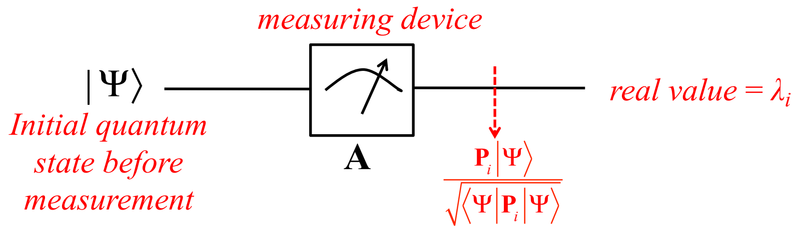
$$|\Psi\rangle \rightarrow \frac{\mathbf{P}_i|\Psi\rangle}{\sqrt{\langle\Psi|\mathbf{P}_i|\Psi\rangle}}$$



58

Measurement "Collapses" the Qubit

- Consider the following Measurement where the Observable is Represented by $\mathbf{A}$ and thus, the qubit Collapses to one of the scaled Eigenvalues of $\mathbf{A}$.



- The following relationships hold:

$$\mathbf{A}=\sum_{i=1}^{n}\lambda_i\mathbf{P}_i=\sum_{i=1}^{n}\lambda_i|e_i\rangle\langle e_i| \qquad |\Psi\rangle\rightarrow\frac{\mathbf{P}_i|\Psi\rangle}{\sqrt{\langle\Psi|\mathbf{P}_i|\Psi\rangle}}$$

- In general, a Projection Operator, $\mathbf{P}_i$, may not yield a normalized projected vector, in which case, the un-normalized projected quantum state is:

$$|\Psi_{UN-NORM}\rangle=\mathbf{P}_i|\Psi\rangle \quad\Rightarrow\quad |\Psi_e\rangle=\frac{|\Psi_{UN-NORM}\rangle}{\sqrt{\langle\Psi|\mathbf{P}_i|\Psi\rangle}}$$

59

Measurement Example

- Assume that Observable $\mathbf{A}$ is formed from the Computational Basis for a Single Qubit:

$$\mathbf{A}=\sum_{i=0}^{n-1}\lambda_i|i\rangle\langle i|=\sum_{i=0}^{1}\lambda_0|0\rangle\langle 0|+\lambda_1|1\rangle\langle 1|=\sum_{i=0}^{1}\lambda_0\mathbf{P}_0+\lambda_1\mathbf{P}_1$$

- Assume Qubit to be Measured is expressed in the Computation Basis as:

$$|\Psi\rangle=a_0|0\rangle+a_1|1\rangle=\sum_{i=0}^{1}a_i|i\rangle$$

- Multiplying both sides by an Eigenbra of Observable $\mathbf{A}$:

$$\langle j|\Psi\rangle=\sum_{i=0}^{1}a_i\langle j|i\rangle=\sum_{i=0}^{1}a_i\delta_{ji}$$

- Using the Sifting property, we observe:

$$\langle j|\Psi\rangle=\sum_{i=0}^{1}a_i\langle j|i\rangle=\sum_{i=0}^{1}a_i\delta_{ji}=a_j$$

- Squaring both sides of this result yields the Probability by Born's rule:

$$|\langle j|\Psi\rangle|^2=\left|\sum_{i=0}^{1}a_i\langle j|i\rangle\right|^2=\left|\sum_{i=0}^{1}a_i\delta_{ji}\right|^2=|a_j|^2$$

$$\mathrm{Prob}\left[|\Psi\rangle\rightarrow|j\rangle\right]=|a_j|^2=|\langle j|\Psi\rangle|^2=\left|\sum_{i=0}^{1}a_i\langle j|i\rangle\right|^2$$

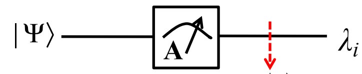
60

Measurement Example (cont.)

- Assume that Observable $\mathbf{A}$ is formed from the Computational Basis for a Single Qubit:

$$|\Psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle = \sum_{i=0}^{1} a_i |i\rangle \qquad \mathbf{A} = \sum_{i=0}^{n-1} \lambda_i |i\rangle\langle i| = \sum_{i=0}^{1} \lambda_0 |0\rangle\langle 0| + \lambda_1 |1\rangle\langle 1| = \sum_{i=0}^{1} \lambda_0 \mathbf{P}_0 + \lambda_1 \mathbf{P}_1$$

- Perform the Projective Measurement wrt to Observable, $\mathbf{A}$:



$$\frac{\mathbf{P}_i|\Psi\rangle}{\sqrt{\langle\Psi|\mathbf{P}_i|\Psi\rangle}}$$

- This results in a measurement outcome of: $\{\lambda_0, \lambda_1\} = \{1, 1\}$
- With the Collapsed state of either:

$$|\Psi\rangle \to \frac{\mathbf{P}_0|\Psi\rangle}{\sqrt{\langle\Psi|\mathbf{P}_0|\Psi\rangle}} = \frac{a_0|0\rangle}{\sqrt{\langle\Psi|0\rangle\langle 0|\Psi\rangle}} = \frac{a_0|0\rangle}{\sqrt{a_0^* a_0}} = \left(\frac{a_0}{|a_0|}\right)|0\rangle \quad \text{with} \quad \mathrm{Prob}\big[|\Psi\rangle \to |0\rangle\big] = |a_0|^2$$

OR

$$|\Psi\rangle \to \frac{\mathbf{P}_1|\Psi\rangle}{\sqrt{\langle\Psi|\mathbf{P}_1|\Psi\rangle}} = \frac{a_1|1\rangle}{\sqrt{\langle\Psi|1\rangle\langle 1|\Psi\rangle}} = \frac{a_1|1\rangle}{\sqrt{a_1^* a_1}} = \left(\frac{a_1}{|a_1|}\right)|1\rangle \quad \text{with} \quad \mathrm{Prob}\big[|\Psi\rangle \to |1\rangle\big] = |a_1|^2$$

61

Second Measurement Example

- Assume that Observable $\mathbf{A_Z}$ is formed as the Pauli-$\mathbf{Z}$ Basis for a Single Qubit:

$$\mathbf{A_Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sum_{i=0}^{1} \lambda_{\mathbf{Z}i} \mathbf{P}_{\mathbf{Z}i} = \lambda_{\mathbf{Z}0}\mathbf{P}_{\mathbf{Z}0} + \lambda_{\mathbf{Z}1}\mathbf{P}_{\mathbf{Z}1} = (1)|0\rangle\langle 0| + (-1)|1\rangle\langle 1|$$

- Since Eigenkets are $|0\rangle$ and $|1\rangle$, the Measurement Basis is the Computational Basis, but Measurement Outcomes are now:

$$\{\lambda_{\mathbf{Z}0}, \lambda_{\mathbf{Z}1}\} = \{+1, -1\}$$

- Thus, this is typically the form of Projective Measurement for a measurements wrt to Computational Basis:

$$|\Psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle = \sum_{i=0}^{1} a_i |i\rangle$$

$$|\Psi\rangle \to \left(\frac{a_0}{|a_0|}\right)|0\rangle \qquad \mathrm{Meas}\big[|\Psi\rangle\big] = \lambda_{\mathbf{Z}0} = +1 \quad \text{with} \quad \mathrm{Prob}\big[|\Psi\rangle \to |0\rangle\big] = |a_0|^2$$

OR

$$|\Psi\rangle \to \left(\frac{a_1}{|a_1|}\right)|1\rangle \qquad \mathrm{Meas}\big[|\Psi\rangle\big] = \lambda_{\mathbf{Z}1} = -1 \quad \text{with} \quad \mathrm{Prob}\big[|\Psi\rangle \to |1\rangle\big] = |a_1|^2$$

62

Third Measurement Example

- Assume that Observable $\mathbf{A_Z}$ is formed as the Pauli-$\mathbf{Z}$ Basis for a Single Qubit:

$$\mathbf{A_Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sum_{i=0}^{1} \lambda_{Zi}\mathbf{P}_{Zi} = \lambda_{Z0}\mathbf{P}_{Z0} + \lambda_{Z1}\mathbf{P}_{Z1} = (1)|0\rangle\langle 0| + (-1)|1\rangle\langle 1|$$

- Since Eigenkets are $|0\rangle$ and $|1\rangle$, the Measurement Basis is the Computational Basis, but Measurement Outcomes are now:

$$\{\lambda_{Z0}, \lambda_{Z1}\} = \{+1, -1\}$$

- Assume that the Quantum State to be Measured is specified in terms of a *<u>different basis than the measurement basis</u>*: $|\Psi\rangle = b_+|+\rangle + b_-|-\rangle$ $\quad |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ $\quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$
- In terms of the Computational Basis, we formulate a change of basis and see that:

$$|\Psi\rangle = b_+|+\rangle + b_-|-\rangle = \left(\frac{b_+ + b_-}{\sqrt{2}}\right)|0\rangle + \left(\frac{b_+ - b_-}{\sqrt{2}}\right)|1\rangle$$

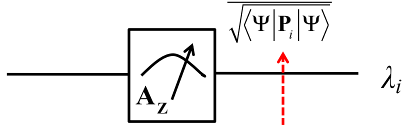
63

Third Measurement Example (cont)

- The Measurement:

$$\frac{\mathbf{P}_i|\Psi\rangle}{\sqrt{\langle\Psi|\mathbf{P}_i|\Psi\rangle}}$$

$|\Psi\rangle = b_+|+\rangle + b_-|-\rangle$



- Measurement Result:

$$|\Psi\rangle \rightarrow \frac{\mathbf{P}_0|\Psi\rangle}{\sqrt{\langle\Psi|\mathbf{P}_0|\Psi\rangle}} = \frac{|0\rangle\langle 0|\Psi\rangle}{\sqrt{\langle\psi|0\rangle\langle 0|\Psi\rangle}} = \frac{\left(\frac{b_+ + b_-}{\sqrt{2}}\right)|0\rangle}{\sqrt{\left(\frac{b_+^* + b_-^*}{\sqrt{2}}\right)\left(\frac{b_+ + b_-}{\sqrt{2}}\right)}} = \frac{b_+ + b_-}{\sqrt{(b_+^* + b_-^*)(b_+ + b_-)}}|0\rangle$$

$\mathrm{Meas}\left[|\Psi\rangle\right] = \lambda_{Z0} = +1$ with $\mathrm{Prob}\left[|\Psi\rangle \rightarrow |0\rangle\right] = \left|\frac{b_+ + b_-}{\sqrt{2}}\right|^2 = \frac{1}{2}(b_+ + b_-)^2$

<u>OR</u>

$$|\Psi\rangle \rightarrow \frac{\mathbf{P}_1|\Psi\rangle}{\sqrt{\langle\Psi|\mathbf{P}_1|\Psi\rangle}} = \frac{|1\rangle\langle 1|\Psi\rangle}{\sqrt{\langle\psi|1\rangle\langle 1|\Psi\rangle}} = \frac{\left(\frac{b_+ - b_-}{\sqrt{2}}\right)|1\rangle}{\sqrt{\left(\frac{b_+^* - b_-^*}{\sqrt{2}}\right)\left(\frac{b_+ - b_-}{\sqrt{2}}\right)}} = \frac{b_+ - b_-}{\sqrt{(b_+^* - b_-^*)(b_+ - b_-)}}|1\rangle$$

$\mathrm{Meas}\left[|\Psi\rangle\right] = \lambda_{Z1} = -1$ with $\mathrm{Prob}\left[|\Psi\rangle \rightarrow |1\rangle\right] = \left|\frac{b_+ - b_-}{\sqrt{2}}\right|^2 = \frac{1}{2}(b_+ - b_-)^2$

64

Projective Measurement Summary

- Measurement Basis is chosen to construct the Observable
  - It spans the Vector Space that contains the Quantum State undergoing measurement
- Measurement Outcome is an eigenvalue of the Observable
  - Observables are Hermitian, thus Eigenvalues (Measurement Outcomes) are Real-valued
- Projective Measurement evolves the Quantum State to one of the Measurement Basis functions with some probability
  - Destroys superposition with respect to the measurement basis, thus it is a destructive measurement
  - This is known as “collapsing” the quantum state to a basis function
- Probability of collapse to a particular Measurement Basis function is calculated according to Born’s Rule
  - Not derivable from Schrodinger’s equation, this is a QM postulate
  - Combines QM deterministic theory with the postulated statistical nature of QM
- Quantum Superposition is Relative to the Observation Basis Set (aka, the Measurement Basis)
  - Basis of Quantum Key Distribution (QKD) methods
  - Basis of one-way or “cluster-state” Quantum Computation

65