Quantum Fourier Transform

Joseph Fourier

$$\mathbf{DFT}_3 = \frac{1}{\left(\sqrt{2}\right)^3}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega^1 & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{bmatrix}$$

Classical Fourier Transform

- Transforms a continuous complex-valued function of one-variable to another continuous complex-valued function of another variable
- Commonly used in signal processing to transform functions of time to functions of frequency
- Expresses a function as a linear combination of a set of oscillatory functions

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\,dt$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{+i\omega t}\,d\omega$$

Discrete Fourier Transform

- The Fourier transform of a discrete function
- As an example, consider a polynomial function of the form:

$$p(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + ... + a_{n-1}x^{n-1}$$

- $p(x)$ can be represented as a column vector, $\mathbf{p}$

$$\mathbf{p} = \begin{bmatrix} a_0 & a_1 & a_2 & ... & a_{n-1} \end{bmatrix}^T$$

Polynomial Evaluation

- Suppose we want to evaluate the polynomial for a set of x-values

$$x_0, \quad x_1, \quad x_2, \quad ..., \quad x_{n-1}$$

- One way to find the set

$$p(x_0), \quad p(x_1), \quad p(x_2), \quad ..., \quad p(x_{n-1})$$

- is to perform a matrix multiplication:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^3 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^3 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^3 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^3 & \cdots & x_k^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^3 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} p(x_0) \\ p(x_1) \\ p(x_2) \\ \vdots \\ p(x_k) \\ \vdots \\ p(x_{n-1}) \end{bmatrix}$$

Polynomial Evaluation

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^3 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^3 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^3 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^3 & \cdots & x_k^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^3 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} p(x_0) \\ p(x_1) \\ p(x_2) \\ \vdots \\ p(x_k) \\ \vdots \\ p(x_{n-1}) \end{bmatrix}$$

- Matrix contains rows that are a geometric series called a Vandermonde matrix Denoted as:

$$\mathbf{V}\left(x_0, x_1, x_2, \ldots, x_{n-1}\right)$$

- For the DFT, the matrix coefficients are powers of the M^{th} roots of unity where the matrix is of size $M \times M$

M^{th} Root of Unity

- For a fixed value of M, there are M different roots of unity:

$$e^{\frac{2\pi i k}{M}} = \omega^k \qquad \omega = e^{\frac{2\pi i}{M}}$$

- The general form of the roots of unity is:

$$\sqrt[M]{1} = \omega^k, \quad k = 1, 2, \ldots, M$$

- As an example, consider:

M	$\omega_k = \omega^{1/k}$	M	$\omega_k = \omega^{1/k}$
1	$e^{\frac{2\pi i 1}{1}}$	4	$e^{\frac{2\pi i 1}{4}}, e^{\frac{2\pi i 2}{4}}, e^{\frac{2\pi i 3}{4}}, e^{\frac{2\pi i 4}{4}}$
2	$e^{\frac{2\pi i 1}{2}}, e^{\frac{2\pi i 2}{2}}$	5	$e^{\frac{2\pi i 1}{5}}, e^{\frac{2\pi i 2}{5}}, e^{\frac{2\pi i 3}{5}}, e^{\frac{2\pi i 4}{5}}, e^{\frac{2\pi i 5}{5}}$
3	$e^{\frac{2\pi i 1}{3}}, e^{\frac{2\pi i 2}{3}}, e^{\frac{2\pi i 3}{3}}$	6	$e^{\frac{2\pi i 1}{6}}, e^{\frac{2\pi i 2}{6}}, e^{\frac{2\pi i 3}{6}}, e^{\frac{2\pi i 4}{6}}, e^{\frac{2\pi i 5}{6}}, e^{\frac{2\pi i 6}{6}},$

Discrete Fourier Transform

- For $M=2^m$, using the Vandermonde matrix and the roots of unity:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & \omega^{1\cdot 1} & \omega^{1\cdot 2} & \cdots & \omega^{1\cdot j} & \cdots & \omega^{1\cdot(M-1)} \\ 1 & \omega^{2\cdot 1} & \omega^{2\cdot 2} & \cdots & \omega^{2\cdot j} & \cdots & \omega^{2\cdot(M-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{k\cdot 1} & \omega^{k\cdot 2} & \cdots & \omega^{k\cdot j} & \cdots & \omega^{k\cdot(M-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{(M-1)\cdot 1} & \omega^{(M-1)\cdot 2} & \cdots & \omega^{(M-1)\cdot j} & \cdots & \omega^{(M-1)\cdot(M-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} p(\omega^0) \\ p(\omega^1) \\ p(\omega^2) \\ \vdots \\ p(\omega^k) \\ \vdots \\ p(\omega^{M-1}) \end{bmatrix}$$

- Matrix is denoted as **DFT**:

$$\mathbf{DFT} = \frac{1}{\sqrt{M}} V\left(\omega^0, \omega^1, \omega^2, \ldots, \omega^{M-1}\right) = \frac{1}{\sqrt{M}}\left[\omega^{jk}\right]$$

DFT is Unitary

$$\mathbf{DFT} = \frac{1}{\sqrt{M}} V\left(\omega^0, \omega^1, \omega^2, \ldots, \omega^{M-1}\right) = \frac{1}{\sqrt{M}}\left[\omega^{jk}\right]$$

- The adjoint of **DFT** is:

$$\mathbf{DFT}^\dagger = \frac{1}{\sqrt{M}}\left[\omega^{kj}\right]^* = \frac{1}{\sqrt{M}}\left[\omega^{-jk}\right]$$

- To show unitary property, multiply:

$$\left[\mathbf{DFT}\right]\left[\mathbf{DFT}^\dagger\right] = \frac{1}{M}\sum_{s=0}^{M-1}\left(\omega^{js}\omega^{-sk}\right) = \frac{1}{M}\sum_{s=0}^{M-1}\omega^{-s(k-j)}$$

- Diagonal Entry of **DFT** occurs when $j=k$:

$$\left[\mathbf{DFT}\right]\left[\mathbf{DFT}^\dagger\right] = \frac{1}{M}\sum_{s=0}^{M-1}\omega^{-s(k-j)} = \frac{1}{M}\sum_{s=0}^{M-1}\omega^{-s(0)} = \frac{1}{M}\sum_{s=0}^{M-1}1 = 1$$

DFT is Unitary

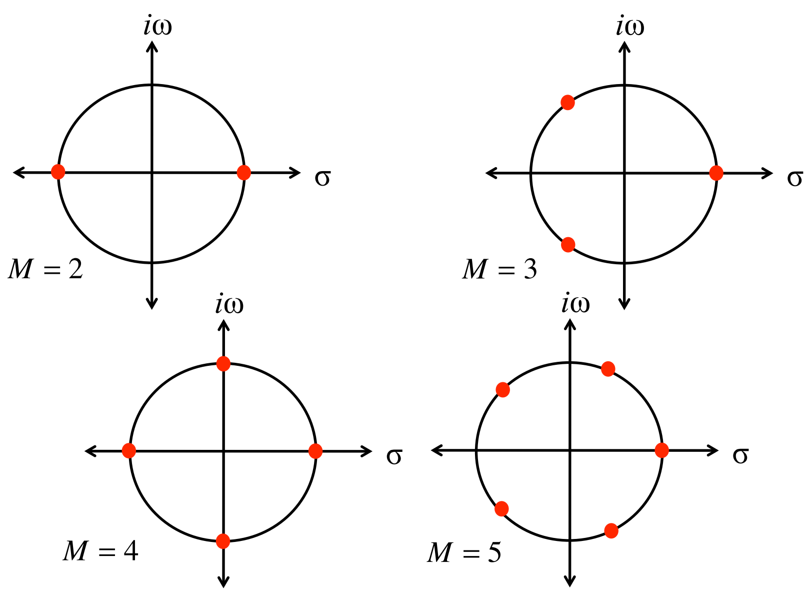
$$\mathbf{DFT} = \frac{1}{\sqrt{M}} V\left(\omega^0, \omega^1, \omega^2, \ldots, \omega^{M-1}\right) = \frac{1}{\sqrt{M}}\left[\omega^{jk}\right]$$

- Off-Diagonal Entry of **DFT** occurs when $j \neq k$
- Leads to a Geometric Progression that sums to Zero
- Therefore DFT is Unitary and:

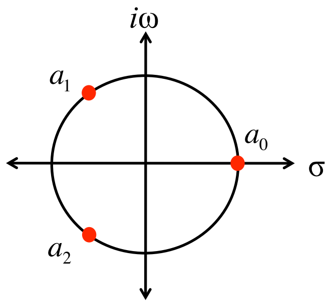
$$[\mathbf{DFT}]\left[\mathbf{DFT}^{\dagger}\right] = \mathbf{I}$$

- DFT matrix evaluates polynomials along equally spaced points on unit circle in complex plane
- Evaluations are periodic since the points go around the circle in a modular fashion

Unit Circle on Complex Plane



Three Roots of Unity



$$a_0 = e^{\frac{2\pi i 3}{3}} = \cos(2\pi) + i\sin(2\pi)$$

$$a_1 = e^{\frac{2\pi i 1}{3}} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$$

$$a_1 = e^{\frac{2\pi i 1}{3}} = \cos(120°) + i\sin(120°)$$

$$a_2 = e^{\frac{2\pi i 2}{3}} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$$

$$a_2 = e^{\frac{2\pi i 2}{3}} = \cos(240°) + i\sin(240°)$$

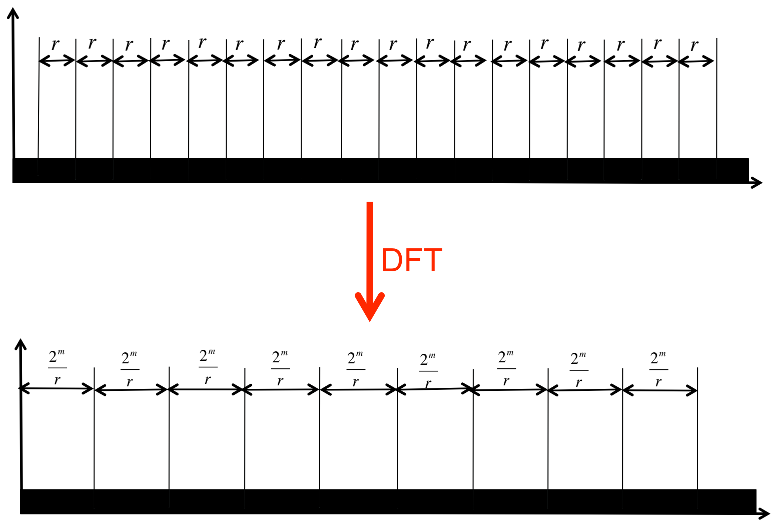
EXAMPLE:

$$(a_2)^3 = \left(e^{\frac{2\pi i 2}{3}}\right)^3 = e^{4\pi i} = \cos(720°) + i\sin(720°) = 1$$

DFT Properties

- Expresses a Function (polynomial) as a Function of (orthogonal) Roots of Unity
- Roots of Unity are Periodic
- If Function is Originally Periodic, DFT is Another Periodic Function with Modified Period
- If Original Function has Period r, Transformed Function has Period $2^m/r$
- Also Eliminates the Offset f the Original Periodic Function

DFT Properties



Quantum Fourier Transform (QFT)

- Variant of DFT Suitable for Implementation as a Quantum Circuit (Algorithm)
- Inverse of QFT is QFT†
- QFT is Crucial Component of Shor's Factoring Algorithm
- QFT is Fast and Composed of "Small" Unitary Operators
- QFT is Linear Operator that Transforms an Orthonormal Basis to Another

QFT

- QFT is Linear Operator that Transforms an Orthonormal Basis

$$\{|0\rangle, |1\rangle, \ldots, |j\rangle, \ldots, |k\rangle, \ldots, |N-1\rangle\}$$

- to the basis:

$$|j\rangle \mapsto \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} e^{\frac{i2\pi jk}{M}} |k\rangle$$

- QFT Transform a Quantum State into Another

$$|v\rangle \mapsto |w\rangle$$

QFT (cont)

$$|v\rangle \mapsto |w\rangle$$

- where,

$$|v\rangle = \sum_{j=0}^{M-1} v_j |j\rangle \qquad |w\rangle = \sum_{k=0}^{M-1} w_k |k\rangle$$

- amplitudes of transformed state w are the DFT of amplitudes of v:

$$w_k = \text{DFT}(v_j) \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} v_j e^{\frac{i2\pi jk}{M}}$$

QFT (cont)

$$|v\rangle \mapsto |w\rangle$$

- when $M=2^n$, Consider the Binary Radix Polynomial Representations of j and k

$$j = j_0 2^{n-1} + j_1 2^{n-2} + \ldots + j_{n-2} 2^1 + j_{n-1} 2^0$$

$$k = k_0 2^{n-1} + k_1 2^{n-2} + \ldots + k_{n-2} 2^1 + k_{n-1} 2^0$$

- QFT can be Rewritten as:

$$|j_0 j_1 \ldots j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}} \sum_{k_0=(0,1)} \sum_{k_1=(0,1)} \ldots \sum_{k_{n-1}=(0,1)} e^{i2\pi j \sum_{m=0}^{n-1} k_m 2^{-m}} |k_0 k_1 \ldots k_m \ldots k_{n-1}\rangle$$

QFT (cont)

- Rearranging and Simplifying:

$$|j_0 j_1 \ldots j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}} \sum_{k_0=(0,1)} \sum_{k_1=(0,1)} \ldots \sum_{k_{n-1}=(0,1)} e^{i2\pi j \sum_{m=0}^{n-1} k_m 2^{-m}} |k_0 k_1 \ldots k_m \ldots k_{n-1}\rangle$$

- yields:

$$|j_0 j_1 \ldots j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}} \sum_{k_0=(0,1)} \sum_{k_1=(0,1)} \ldots \sum_{k_{n-1}=(0,1)} \bigotimes_{m=0}^{n-1} e^{i2\pi j k_m 2^{-m}} |k_m\rangle$$

$$|j_0 j_1 \ldots j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}} \bigotimes_{m=0}^{n-1} \left\{ \sum_{k_m=(0,1)} e^{i2\pi j k_m 2^{-m}} |k_m\rangle \right\}$$

- k_m may only take values $\{0,1\}$:

$$|j_0 j_1 \ldots j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}} \bigotimes_{m=0}^{n-1} \left\{ |0\rangle + e^{i2\pi j 2^{-m}} |1\rangle \right\}$$

QFT (cont)

- QFT:

$$|j_0 j_1 ... j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}} \bigotimes_{m=0}^{n-1} \left\{ |0\rangle + e^{i2\pi j 2^{-m}} |1\rangle \right\}$$

- Simplifying:

$$|0\rangle + e^{i2\pi j 2^{-m}} |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{i2\pi j 2^{-m}} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{i2\pi j 2^{-m}} \end{bmatrix}$$

- Transformation on input $|j\rangle$ Rewritten as:

$$|j\rangle \mapsto \frac{1}{2^{n/2}} \begin{bmatrix} 1 \\ e^{i2\pi(j/2^0)} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ e^{i2\pi(j/2^1)} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ e^{i2\pi(j/2^2)} \end{bmatrix} \otimes ... \otimes \begin{bmatrix} 1 \\ e^{i2\pi(j/2^{n-1})} \end{bmatrix}$$

3-Qubit QFT Example

- QFT of Basis Vector $|000\rangle$ where $n=3$ and $j=0$:

$$|000\rangle \mapsto \frac{1}{2^{3/2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\left(\sqrt{2}\right)^3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$|000\rangle \mapsto \frac{1}{2^{3/2}} [|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle]$$

3-Qubit QFT Example (cont)

- QFT of Basis Vector $|001\rangle$ where n=3 and j=1:

$$|001\rangle \mapsto \frac{1}{2^{3/2}}\begin{bmatrix}1\\1\end{bmatrix}\otimes\begin{bmatrix}1\\e^{i2\pi(1/2)}\end{bmatrix}\otimes\begin{bmatrix}1\\e^{i2\pi(1/4)}\end{bmatrix}$$

$$=\frac{1}{2^{3/2}}\begin{bmatrix}1\\1\end{bmatrix}\otimes\begin{bmatrix}1\\-1\end{bmatrix}\otimes\begin{bmatrix}1\\i\end{bmatrix}$$

$$|001\rangle \mapsto \frac{1}{2^{3/2}}\begin{bmatrix}1\\i\\-1\\-i\\1\\i\\-1\\-i\end{bmatrix}$$

$$|001\rangle \mapsto \frac{1}{2^{3/2}}\left[|000\rangle-|010\rangle+|100\rangle-|110\rangle+i\left(|001\rangle-|011\rangle+|101\rangle-|111\rangle\right)\right]$$

QFT Quantum Cascade

- Recall: $|j_0 j_1 \ldots j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}}\bigotimes_{m=0}^{n-1}\left\{|0\rangle + e^{i2\pi j 2^{-m}}|1\rangle\right\}$
- Can Use Regular Multiplication of ket Vectors

$$\bigotimes_{m=0}^{n-1}\left\{|0\rangle + e^{i2\pi j 2^{-m}}|1\rangle\right\} = \prod_{m=0}^{n-1}\left(|0\rangle + e^{i2\pi j 2^{-m}}|1\rangle\right)$$

$$|j_0 j_1 \ldots j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}}\left(|0\rangle + e^{i2\pi j 2^{0}}|1\rangle\right)\left(|0\rangle + e^{i2\pi j 2^{1}}|1\rangle\right)\ldots\left(|0\rangle + e^{i2\pi j 2^{n-1}}|1\rangle\right)$$

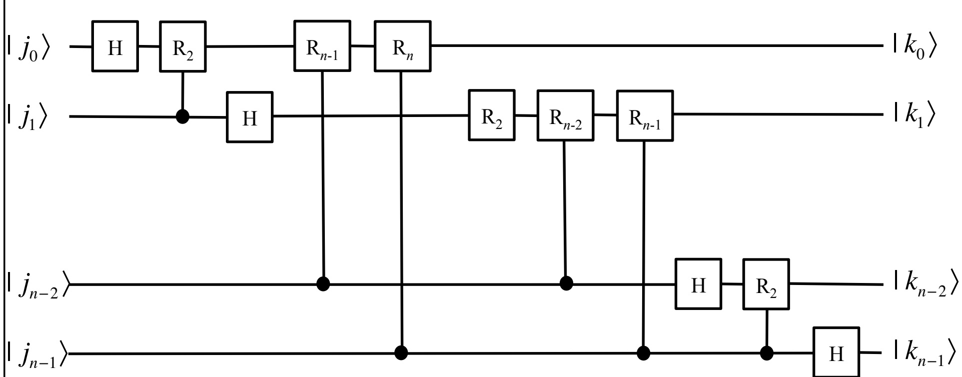
- Allows a Cascade to be Realized Using Hadamard Gates and Controlled-Rotation Gates
- Basis Vector Placed in Superposition using Hadamard First then Undergoes Rotations

QFT Quantum Cascade (circuit)

- Recall R_k Gate
- Transforms a Qubit by Multiplying its Projection on $|1\rangle$ by $e^{i2\pi/2^k}$

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^k} \end{bmatrix}$$

QFT Quantum Cascade (circuit)



3-Qubit QFT Example

- Recall the Identities: $e^{i\theta} = \cos\theta + i\sin\theta$

$$\cos(2k+1)\frac{\pi}{2} = 0 \qquad \sin(2k+1)\frac{\pi}{2} = 1$$

$$e^{i\pi/2} = i \qquad e^{i\pi/4} = \sqrt{i}$$

- Use the Following:

$$\omega = \sqrt{i} \qquad \omega^2 = i \qquad \omega^4 = -1$$

- Rotation Gates Become:

$$R_2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \omega^2 \end{bmatrix}$$

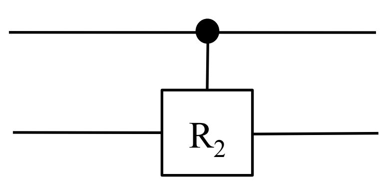
$$R_3 = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}$$

Transfer Matrix – Controlled R^2

- Single Qubit R_2 Transfer Matrix:

$$R_2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \omega^2 \end{bmatrix}$$

- We want a Controlled Gate:



- The Transformations are:

$$|00\rangle \mapsto |00\rangle \qquad |10\rangle \mapsto |10\rangle$$

$$|01\rangle \mapsto |01\rangle \qquad |11\rangle \mapsto e^{i2\pi/2^2}\,|11\rangle$$

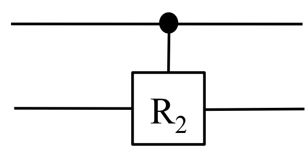
Transfer Matrix – Controlled R^2

- Controlled-R_2 Transfer Matrix using Projections:

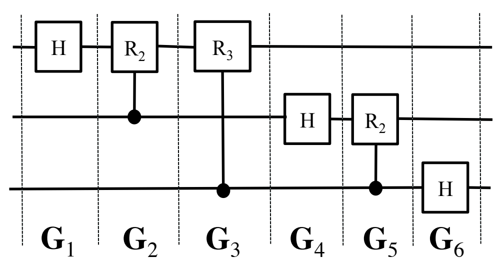
$$\mathbf{G}_{R_2} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + e^{i2\pi/2^2}|11\rangle\langle 11|$$

$$\mathbf{G}_{R_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i2\pi/2^2} \end{bmatrix}$$

$$\mathbf{G}_{R_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i2\pi/2^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix}$$



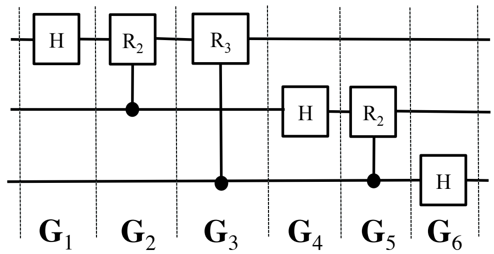
QFT Quantum Cascade



$$\mathbf{G}_1 = \mathbf{H} \otimes \mathbf{I} \otimes \mathbf{I} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

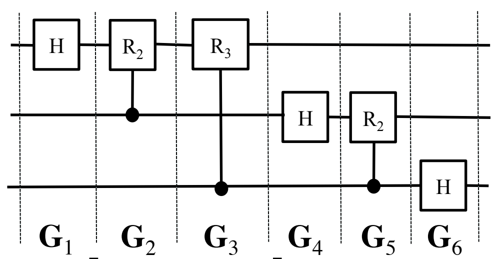
$$\mathbf{G}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

QFT Quantum Cascade (circuit)



$$\mathbf{G}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{I}_4 & \mathbf{I}_4 \\ \mathbf{I}_4 & -\mathbf{I}_4 \end{bmatrix}, \qquad \mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

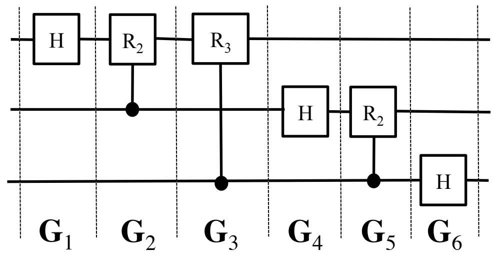
QFT Quantum Cascade (circuit)



$$\mathbf{G}_2 = \mathbf{G}_{R_2} \otimes \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_4 & 0 \\ 0 & \mathbf{M}_{(2,4)} \end{bmatrix}$$

$$\mathbf{M}_{(2,4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix}$$

QFT Quantum Cascade (circuit)

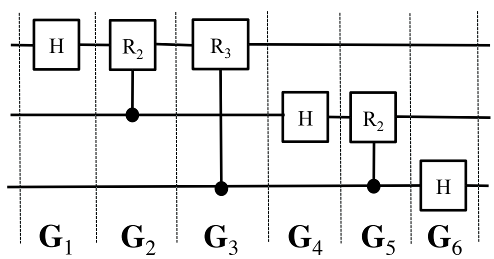


$$|000\rangle \mapsto |000\rangle,\ \ |001\rangle \mapsto |001\rangle,\ \ |010\rangle \mapsto |010\rangle,\ \ |011\rangle \mapsto |011\rangle$$

$$|100\rangle \mapsto |100\rangle,\ \ |101\rangle \mapsto \omega|101\rangle,\ \ |110\rangle \mapsto |110\rangle,\ \ |111\rangle \mapsto \omega|111\rangle$$

$$\mathbf{G}_3 = \begin{bmatrix} \mathbf{I}_4 & 0 \\ 0 & \mathbf{M}_{(3,4)} \end{bmatrix} \qquad \mathbf{M}_{(3,4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{bmatrix}$$

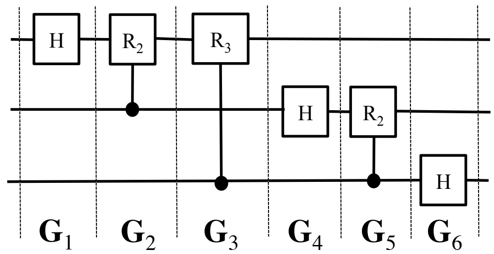
QFT Quantum Cascade (circuit)



$$\mathbf{G}_4 = \mathbf{I} \otimes \mathbf{H} \otimes \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

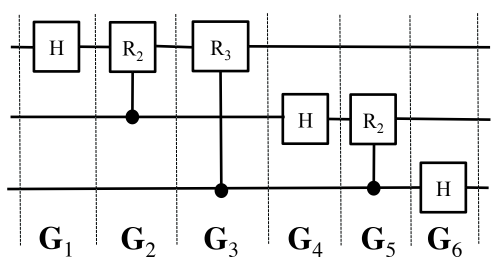
$$\mathbf{G}_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

QFT Quantum Cascade (circuit)



$$\mathbf{G}_4 = \begin{bmatrix} \mathbf{M}_{(4,4)} & 0 \\ 0 & \mathbf{M}_{(4,4)} \end{bmatrix} \qquad \mathbf{M}_{(4,4)} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

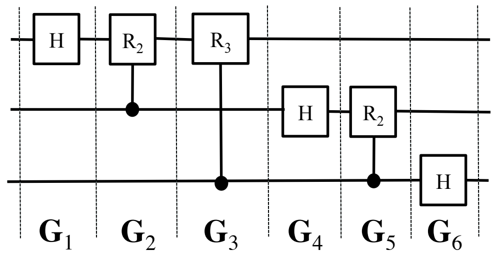
QFT Quantum Cascade (circuit)



$$\mathbf{G}_5 = \mathbf{I} \otimes \mathbf{G}_{R_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{(5,4)} & 0 \\ 0 & \mathbf{M}_{(5,4)} \end{bmatrix}$$

$$\mathbf{M}_{(5,4)} = \mathbf{G}_{R_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix}$$

QFT Quantum Cascade (circuit)



$$\mathbf{G}_6 = \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{M}_{(6,4)} & 0 \\ 0 & \mathbf{M}_{(6,4)} \end{bmatrix}$$

$$\mathbf{M}_{(6,4)} = \mathbf{G}_{R_2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

3-Qubit QFT

- General Transformation is:

$$| w\rangle = \mathbf{G}_6\mathbf{G}_5\mathbf{G}_4\mathbf{G}_3\mathbf{G}_2\mathbf{G}_1 | v\rangle$$

$$\mathbf{G}_t = \mathbf{G}_6\mathbf{G}_5\mathbf{G}_4\mathbf{G}_3\mathbf{G}_2\mathbf{G}_1$$

$$\mathbf{G}_t = \frac{1}{\left(\sqrt{2}\right)^3} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \omega^2 & -1 & -\omega^2 & 1 & \omega^2 & -1 & -\omega^2 \\ 1 & -\omega^2 & -1 & \omega^2 & 1 & -\omega^2 & -1 & \omega^2 \\ 1 & \omega^1 & \omega^2 & \omega^3 & -1 & -\omega^1 & -\omega^2 & -\omega^3 \\ 1 & -\omega^1 & \omega^2 & -\omega^3 & -1 & \omega^1 & -\omega^2 & \omega^3 \\ 1 & \omega^3 & -\omega^2 & -\omega^5 & -1 & -\omega^3 & \omega^2 & \omega^5 \\ 1 & -\omega^3 & -\omega^2 & \omega^5 & -1 & \omega^3 & \omega^2 & -\omega^5 \end{bmatrix}$$

3-Qubit QFT

- Note that: $-1 = e^{i2\pi/2} = \omega^4$

$$-\omega^2 = \omega^6 \quad -\omega^1 = \omega^5 \quad -\omega^3 = \omega^7$$

$$\mathbf{G}_t = \frac{1}{\left(\sqrt{2}\right)^3}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega^1 & \omega^6 & \omega^3 \\ 1 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{bmatrix}$$

3-Qubit QFT

- Consider the Permutation Matrix:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Application of Permutation Matrix to $\mathbf{G}_t$ Yields the QFT_3 (DFT_3) Matrix

3-Qubit QFT

- DFT Matrix Results:

$$\mathbf{QFT}_3 = \mathbf{PG}_t = \frac{1}{\left(\sqrt{2}\right)^3}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega^1 & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{bmatrix}$$