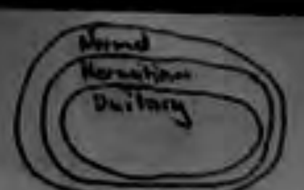
Normal: $NN^\dagger = N^\dagger N$
Hermitian: $H^\dagger = H$
Unitary: $U^{-1} = U^\dagger$



Show Unitary matrices are Hermitian:

$U^\dagger = U^{-1}$ (1)

$(U^\dagger)^{-1} = (U^{-1})^{-1} = U$ (2)

mult. both sides of (1) by U:

$UU^\dagger = UU^{-1} = I$

$U^\dagger U = U^{-1}U = I$

$\therefore UU^\dagger = U^\dagger U$ is normal (3)

mult. both sides of (3) by U^{-1}:

$\underline{U^{-1}}U^\dagger U = \underline{U^{-1}}U^{-1}U$

$U^{-1}U^\dagger U = U^{-1}\underline{U^\dagger U}$

$U^{-1}UU^\dagger = U^{-1}\underline{UU^\dagger}$

$U^{-1}UU^\dagger = UU^{-1}U^\dagger$

$U^{-1}UU^\dagger = UU^{-1}U$

$IU^\dagger = IU$

$U^\dagger = U$

~~Substitute (2) into (3)~~

$\underline{U^{-1}}U^\dagger U = \underline{U^{-1}}U^{-1}U$

LHS: $U^\dagger = U$:

$U^{-1}UU = U^{-1}U^{-1}U$

$IU = U^{-1}U^{-1}U$

$U = U^{-1}U^{-1}U$

RHS: $U^{-1}U = UU^{-1} = I$

$U = U^{-1}\underline{UU^{-1}}$

$U = U^{-1}I$

$U = U^{-1}$

RHS: since unitary $U^{-1} = U^\dagger$

$U = U^\dagger$ □

Power Series of Matrix Exponential (A)

$$e^{\beta A} = \sum_{k=0}^{\infty} \frac{(\beta)^k}{k!} A^k$$

When A is Hermitian and $AA = A^2 = I$:

$$e^{\beta A} = \sum_{k=0}^{\infty} \frac{(\beta)^k}{k!} A^k = I + \beta A + \frac{\beta^2}{2!} I + \frac{\beta^3}{3!} A + \dots$$

$$= \left[1 + \frac{\beta^2}{2!} + \frac{\beta^4}{4!} + \dots\right] I + \left[\beta + \frac{\beta^3}{3!} + \frac{\beta^5}{5!} + \dots\right] A \quad (4)$$

Power series for exponential of $\beta \in \mathbb{R}$

$$e^{\beta} = \sum_{k=0}^{\infty} \frac{\beta^k}{k!} = 1 + \frac{\beta}{1!} + \frac{\beta^2}{2!} + \frac{\beta^3}{3!} + \dots + \frac{\beta^n}{n!} + \dots$$

$$e^{-\beta} = \sum_{k=0}^{\infty} \frac{(-\beta^k)}{k!} = 1 - \frac{\beta}{1!} + \frac{\beta^2}{2!} - \frac{\beta^3}{3!} + \dots + (-1)^n \frac{\beta^n}{n!} + \dots$$

Def. of hyperbolic cosine of $\beta \in \mathbb{R}$:

$$\cosh(\beta) = \frac{e^{\beta} + e^{-\beta}}{2} = \frac{1}{2}\left(1 + \frac{\beta}{1!} + \frac{\beta^2}{2!} + \frac{\beta^3}{3!} + \dots\right) + \frac{1}{2}\left(1 - \frac{\beta}{1!} + \frac{\beta^2}{2!} - \frac{\beta^3}{3!} + \dots\right)$$

$$= 1 + \frac{\beta^2}{2!} + \frac{\beta^4}{4!} + \dots$$

Def. of hyperbolic sine of $\beta \in \mathbb{R}$

$$\sinh(\beta) = \frac{e^{\beta} - e^{-\beta}}{2} = \frac{\beta}{1!} + \frac{\beta^3}{3!} + \frac{\beta^5}{5!} + \dots$$

Plugging into expansion (4) for $e^{\beta A}$ above:

$$e^{\beta A} = \cosh(\beta) I + \sinh(\beta) A \quad (5)$$

Change $e^{\beta A}$ to $e^{i\beta A}$:

$$e^{i\beta A} = \cosh(i\beta) I + \sinh(i\beta) A \quad (6)$$

$\cosh(i\beta) = \frac{1}{2}(e^{i\beta} + e^{-i\beta})$

$\sinh(i\beta) = \frac{1}{2}(e^{i\beta} - e^{-i\beta})$

Euler's relation:

$e^{i\beta} = \cos\beta + i\sin\beta$ $\quad e^{-i\beta} = \cos\beta - i\sin\beta$

$\sinh(i\beta) = \frac{1}{2}[\cos\beta + i\sin\beta - \cos\beta + i\sin\beta] = i\sin\beta$

$\cosh(i\beta) = \frac{1}{2}[\cos\beta + i\sin\beta + \cos\beta - i\sin\beta] = \cos\beta$

Substituting into (6) yields:

$$\boxed{e^{i\beta A} = \cos(\beta) I + i\sin(\beta) A}$$