Introduction to Quantum Computation Reliability

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***Abstract*—An overview of the quantum computing paradigm with a focus on reliability is provided in a tutorial form intended for practitioners and researchers in the test and reliability community. It is assumed that readers have little prior knowledge of quantum informatics. An introductory description of the mathematical models of a qubit and quantum information processing operations and projective measurement is presented as background. Discussions of quantum error sources and associated fault models are included using the concepts and notation explained in the background section. Topics related to the decoherence problem are also included. The concept of a logical qubit and how quantum error detection and correction can be applied to enhance reliability of quantum computations is formulated using the notions of classical fault models and error detection and correction techniques. The general concept of quantum error detection and correction as applied to enhancing quantum computational reliability is discussed including an example of one of the first such methods originally introduced in 1995.**

Keywords—quantum computing, quantum fault models, logical qubit, reliability, quantum error correction

I. INTRODUCTION

The theoretical concept of a quantum computer (QC) and quantum computation in general is usually attributed to a presentation and accompanying 1982 paper by physicist Richard Feynman when he first coined the term and provided an outline of possible computational operations [1]. Acknowledgement considering the origin of quantum computers must also be given to Paul Benioff in his consideration of the quantum mechanical model of Turing machines in his 1980 paper [2]. Feynman was concerned with the topic of physics simulations and poses the question "... how can we simulate quantum mechanics?" Feynman then answers this question by stating "Let the computer itself be built of quantum mechanical elements which obey quantum mechanical laws" and follows with a section titled "Quantum Computers – Universal Quantum Simulators." Alternatively, Benioff proposed and explored the concept of a series of quantum state evolutions as mapped to a corresponding series of state transitions in the classical Turing model of computation.

While Benioff and Feynman can be credited with formulating the initial concepts of a quantum computer, one of the first formal definitions of a complete quantum computational paradigm is due to Deutsch in his paper that appeared three years after Feynman's paper [3]. Deutsch explained the concept of "quantum parallelism" that enabled QC speedup due to the fact that a given computation can be performed over a set of data that is simultaneously represented through the exploitation of quantum superposition. The Deutsch computational paradigm consists of three requirements in terms of implementation; (i) the ability to initialize the information-carrying hosts to a known value, (ii) the ability to perform transformations of the data values after initialization, and (iii) the final step of measuring or observing the transformed information. Due to some of the similarities of Deutsch's 1985 computational paradigm to that of conventional electronic digital computation, the Deutsch model of computation is often referred to as the "gate" model for QC. In considering the gate model of QC, intermediate transformations are analogous to the operation of a sequence of discrete logic gates or assembly instructions that transform subsets of the quantum information comprising the processing task.

The fundamental unit of information in QCs is the quantum bit or "qubit." The term "qubit" appeared in a paper that is concerned with the subject of quantum information theory where a theorem analogous with Shannon's noiseless coding theorem is developed for quantum information [4]. In contrast with a conventional bit that is modeled as taking on the value of an integer from the set, $\mathbb{B} = \{0,1\}$, a qubit is typically modeled as an element from a complex two-dimensional Hilbert vector space, $\mathbb{H}_2 = (\mathcal{V}, \mathbb{C}, \langle|\rangle)$ where $|v_i\rangle \in \mathcal{V}$ is a two-dimensional column vector in the group $\mathcal{V}$ with a group operator of vector addition, $\mathbb{C}$ is the field of complex scalars, and $\langle|\rangle$ denotes the inner product operation over two elements of $\mathcal{V}$.

The QC community uses the "braket" notation of quantum physics due to Dirac [5] where $|v_i\rangle$ represents a column-vector, $\langle v_i|$ a row vector, $\langle v_i|w_j\rangle$ the inner product, $|v_i\rangle\langle w_j|$ the outer product, and $\langle v_i\rangle$ the expected value. It is noted that another common and more general mathematical model of the qubit is the 2×2 density matrix and it is equally valid. The density matrix model is more general since it allows for a single qubit to be comprised of a statistical ensemble of pure states whereas the column vector model assumes the qubit is in the form of a single pure state. The density matrix representation, ρ, of a qubit in a pure state, $|v_i\rangle$, is given as $\rho = |v_i\rangle\langle v_i|$ whereas the more general case where a qubit is represented as a statistical ensemble of k different pure states is given by $\rho = \sum_{i=1}^{k} p_i |v_i\rangle\langle v_i|$ where p_i is the objective probability that the qubit can be in the pure state $|v_i\rangle$.

In terms of implementation, the physical representation of a qubit is an observable or measurable characteristic of a particle or quasi-particle. Examples include the spin, location, and

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energy levels attributed to fermionic particles such as electrons, the polarization, frequency, and spatial location of bosonic particles such as photons, or the energy levels of an ion. Quasi-particles are objects that behave or exhibit measurable quantum mechanical characteristics but are not actual particles. Examples of quasi-particles include quantum dots, diamond-nitrogen vacancies, 2-dimensional anyons, and even macroscopic circuits that behave as quantum oscillators such as superconducting semiconductor circuits containing Josephson junctions. When macroscopic objects exhibit microscopic quantum behavior, they are referred to as mesoscopic qubits. An important concept regarding physical implementations of qubits is that, it is not the particle or quasi-particle that serves as the qubit, rather it is an observable characteristic of the particle that serves to represent quantum information. The particle or quasi-particle is simply the "host" information carrier.

There are several different computational paradigms under consideration by the QC community both in terms of the underlying computational paradigm as well as the physical technology for representing qubits and associated operations that implement their transformation or processing. These different paradigms can be considered as either modifications to the Deutsch gate model or as paradigms with substantially different approaches. A common motivating factor for these alternative QC approaches is to exploit properties of the technology to enhance certain aspects including reliability, scalability, or both. We restrict our focus in this paper to concentrate mainly on the gate model of computation since it is currently one of the more prevalent and popular approaches at the time of this writing.

The reason that Feynman predicted that conventional computers were insufficient for simulating quantum phenomena is due to the very non-intuitive behavior of atomic-scale particles as postulated by the theories of quantum mechanics and quantum electrodynamics. These phenomena include quantum superposition, uncertainty principles, entanglement, measurement, and others. While we will briefly explore some of these properties in a following section with respect to their influence in QC reliability, the concept of quantum superposition and measurement is useful here to describe one of the major challenges of QC, that of decoherence.

As an introduction, quantum superposition refers to the ability of a qubit to exist in a linear combination of basis states. In the gate-model of computation, quantum superposition is a key element in providing the quantum parallelism as explained by Deutsch. The computational advantage arises from the fact that a group of qubits in superposition can simultaneously represent many different values. Thus, if a computation is viewed as a function to be evaluated over a set of data values, that function need only be computed a single time when the input values are represented by a group of superimposed qubits. In contrast, a Turing machine would require that the function be evaluated individually for each of the values in the set of interest. Speed-up of such a computation in a Turing model can only be achieved by spatial parallelism wherein multiple Turing machines execute concurrently with different function arguments, or, with temporal parallelism such as a pipelined Turing machine. While quantum parallelism is a great advantage of QC, it does require that the qubits maintain their state of superposition during the computation, or, a coherent state. Furthermore, a quantum observation or measurement causes the superimposed state of the qubits to "collapse" into a basis value that is one of the superimposed states with respect to some set of basis vectors known as the "measurement basis." An unintentional measurement during a quantum computation can cause the advantage of quantum parallelism to be lost and extreme measures must be taken to prevent this from occurring. Another interesting phenomenon is that the measuring device used to make the observation defines the basis set with respect to how the superimposed qubits collapse. That is, the measurement device affects which possible outcomes result from a set of qubits in superposition.

Because the host particles adhere to the postulates of quantum mechanics, small external disturbances or influences from the external physical environment can inadvertently change the quantum state and can result in the occurrence of an unintentional measurement. When a group of qubits suffer from such an unintentional measurement, they are said to "decohere." Decoherence is one of the biggest challenges in QC and is a key factor in considering QC reliability. Maintaining the coherence of a set of qubits requires careful control of the environment to prevent external influences from affecting the computation and such control can be a challenging requirement. In fact, it is generally an accepted fact that prevention of decoherence will never be entirely achieved with only environmental isolation mechanisms. Thus, the QC community is motivated to devise methods for reliable quantum computation in an environment where there is always a non-zero probability of decoherence.

The prevention of decoherence poses a tradeoff in architectural and implementation considerations for a QC. The tradeoff comes about because it is necessary to have the ability for qubits to interact with one another so that conditional computations can be implemented. As an example of such an intentional interaction, it is necessary that the value of one qubit can conditionally affect the type of computation performed on another qubit. However, enabling such interactions also means that decoherence events from the environment are likewise possible and can thus also affect the value of the qubits in the form of a decoherence event. The tradeoff manifests in that implementations that provide for ease in conditional multi-qubit operations similarly provide ease for the external environment to cause decoherence events. In terms of technology, the use of quantum photonics is a good example of this tradeoff. The fact that interactions among bosons are difficult to achieve means that quantum computations using photons as host particles are generally less susceptible to decoherence events as compared to other host particles that are more sensitive to environmental influences; however, this same phenomenon is also the reason that it is difficult to implement high-fidelity QC gates requiring multi-qubit interaction using quantum photonic technology.

Reliability in quantum computation is a big issue due to the inherent fragility of a qubit with respect to maintaining a state of coherence. As is the case with electronic technology, QC reliability is likewise related to the particular form of implementation technology being used, as just mentioned with respect to the tradeoff example for quantum photonic technology. Decoherence and other sources of error provide the foundation for formulating fault models of quantum computing and these will be explored in more detail.

Given a set of applicable fault models, methods to enhance QC reliability can be explored and are currently an active area of research. The first general purpose quantum computers have begun to be available for use in the past few years as of this writing. Because these QC rely on the use of physical qubits that are generally susceptible to decoherence and other errors, they are considered to be "noisy" quantum computers. Furthermore, because this class of QC are comprised of tens to hundreds of qubits, they are also considered to be predecessors to "intermediate scale" computers since it is recognized that practical QC will likely require a significantly increased number of qubits to tackle many practical processing jobs of interest. For this reason, the QC community considers the current and immediate future-generations of QC to be characterized as "noisy intermediate-scale quantum computers" (NISQ) as coined by John Preskill in a presentation he gave in 2017 [6]. Due in large part to the decoherence tradeoff previously described, the QC community is in general agreement that the next generation of QC beyond the NISQ era will require the use of quantum error correction to enhance reliability as the number of qubits increases. These next-generation QC will comprise "logical qubits" that are internally represented by a number of noisy physical qubits. The use of multiple physical qubits to comprise a single logical qubit will provide the redundancy that is a well-known fundamental requirement for error detection and correction (EDAC) techniques. A very active area of current research is the quest for finding such quantum EDAC frameworks that will enable the next-generation of QC with acceptable levels of reliability as well as scalability.

The remainder of this introductory paper will provide basic background on the gate-model of QC to use as an example for reliability enhancement issues. We will discuss quantum technology fault models, their basis in terms of error sources, and how they influence and guide quantum EDAC approaches. Additionally, we will describe an early quantum EDAC approach to show how it can enhance QC reliability.

II. Quantum Computation Background

A. General Mathematical Model of a Qubit and Observables

From a mathematical interpretation, a pure state qubit is an element of a two-dimensional Hilbert vector space, $\mathbb{H}_2$, and can be represented as

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle. \tag{1}$$

The scalar coefficients, $\alpha, \beta \in \mathbb{C}$, are known as the "probability amplitudes" of $|\Psi\rangle$. The basis vectors $|0\rangle, |1\rangle \in \mathcal{V}$ span $\mathbb{H}_2$ and are referred to as the "computational basis." Explicitly, the computational basis is given as

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}, |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}. \tag{2}$$

Thus, the general form of qubit $|\Psi\rangle$ is likewise explicitly written as

$$|\Psi\rangle = \begin{bmatrix}\alpha\\\beta\end{bmatrix}. \tag{3}$$

Furthermore, due to a postulate of quantum mechanics known as "Born's rule," [7] the following relationship must hold regarding the "probability amplitudes" of the qubit $|\Psi\rangle$,

$$|\alpha|^2 + |\beta|^2 = 1. \tag{4}$$

When either α or β are zero-valued, the qubit is said to be in a basis state, otherwise it is in a state of quantum superposition with respect to the computational basis. Measurements of the state of a qubit are accomplished using observables that are modeled as Hermitian matrices formed as a sum of projection matrices formed from a set of basis vectors. Thus, if a qubit is measured with respect to an observable formed from the computational basis, the result of the measurement is a scalar that is one of the eigenvalues of the observable. Likewise, the measurement will force the quantum state to "collapse" into one of the eigenvectors of the observable that corresponds to the resulting eigenvalue of the observable. If the observable is formed using the computational basis, then the measurement will cause the quantum state to collapse into either $|0\rangle$ or $|1\rangle$. Because the measurement causes the quantum state to transition to a basis vector that is one of the observable eigenvectors, the act of performing the measurement will cause the qubit to lose superposition with respect to the measurement basis. This is known as a "projective measurement" since the observable is a sum of projection matrices formed from the measurement basis set. Due to Born's rule, the result of the measurement is probabilistic in nature. The probabilities that the projective measurement of $|\Psi\rangle$ causes collapse into $|0\rangle$ or $|1\rangle$, when the observable is constructed from the computational basis, is

$$P[|\Psi\rangle \to |0\rangle] = |\alpha|^2 \tag{5}$$

$$P[|\Psi\rangle \to |1\rangle] = |\beta|^2. \tag{6}$$

It should be noted that this projective measurement does not include the possibility that an unintentional measurement due to external environmental influences, or decoherence, occurs. There are other models for measurement observables that include this possibility such as the "positive operator-valued measure" (POVM) that are commonly used.

Due to Born's rule, the norm of a pure state qubit is always unity and thus qubits can more accurately be described as unity-magnitude vectors, or rays, in $\mathbb{H}_2$. Due to the complex-valued probability amplitudes, an alternative model of a qubit is

$$|\Psi\rangle = e^{i\gamma}\left[cos\left(\tfrac{\theta}{2}\right)|0\rangle + e^{i\phi}sin\left(\tfrac{\theta}{2}\right)|1\rangle\right], \tag{7}$$

where,

$$\alpha = e^{i\gamma}cos\left(\tfrac{\theta}{2}\right), \tag{8}$$

$$\beta = e^{i\gamma}e^{i\phi}sin\left(\tfrac{\theta}{2}\right). \tag{9}$$

In this form, γ is a global phase angle since it affects both basis vector components and ϕ is the relative phase angle since it represents the phase of the $|1\rangle$ component with respect to that of the $|0\rangle$ component. Since the information content of a qubit only depends upon the values of the probability amplitudes of each basis vector component and their relative phase difference, the common multiplicative factor $e^{i\gamma}$ is irrelevant. When a qubit is considered in the form of Equations (1) or (7), the only parameters that affect the information content are (α, β) or alternatively, (ϕ, θ) since the global phase, γ, does not affect the relative phase difference between α and β.

Given these models of a qubit, a geometric interpretation is useful and is known as the "Bloch sphere." The Bloch sphere model is one where a pure state qubit is considered to be a point on the surface of a unit sphere centered at the origin. In spherical coordinates, the magnitude is not considered since it is always unity by Born's rule. The relative phase ϕ represents the angle in a plane parallel to the *x*-*y* plane that is relative to the *x*-axis. The angle θ represents the angle in a plane perpendicular to the *x*-*y* plane and relative to the *z*-axis. The transformations from the spherical to the rectangular coordinate model of a qubit are given by Equations (8) and (9).

B. Quantum Computation

A quantum computation applied to a qubit is modeled as a linear transformation of the qubit vector and is thus defined by a linear transformation matrix with complex components. Since Born's rule must always hold, and due to the fact that a qubit is physically a wave function solution of Schrödinger's equation, all such transformation matrices are unitary meaning that their inverses are simply their conjugate transpose. That is, a computation is represented by a unitary matrix **U** satisfying $\mathbf{U}^{-1} = \mathbf{U}^{\dagger} = (\mathbf{U}^{T})^{*}$. Using the geometric interpretation of Bloch's sphere, unitary transformations can be considered as rotation matrices that move the qubit position from one point on the sphere surface to another.

Some single-qubit operations are well known and have specific names. For example, the Pauli operators denoted as **X** (or "*NOT*"), **Y**, and **Z** operations represent 180° rotations about the *x*-, *y*-, and *z*-axes respectively. Some common single-qubit operations only affect the relative phase angle, ϕ, such as the **Z**, **S**, and **T** operations that rotate the qubit in a plane parallel to the *x*-*y* plane by 180°, 90°, and 45° respectively. Other common single-qubit operations modify the probability amplitudes such as the Hadamard, **H**, and the "square-root-of-NOT", **V**. These are given in matrix form as

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix},$$
$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}, \mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \mathbf{V} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}. \quad (10)$$

Multi-qubit operations are represented as transformations over the composite quantum state vector formed as the outer product of the single-qubit pure-state vectors. As an example, two qubits $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\Phi\rangle = \delta|0\rangle + \sigma|1\rangle$ represent an overall quantum state as shown in Equation (11).

$$|\Psi\Phi\rangle = |\Psi\rangle \otimes |\Phi\rangle = (\alpha|0\rangle + \beta|1\rangle)(\delta|0\rangle + \sigma|1\rangle)$$
$$= \alpha\delta|00\rangle + \alpha\sigma|01\rangle + \beta\delta|10\rangle + \beta\sigma|11\rangle \quad (11)$$

A multi-qubit unitary transformation matrix is of dimension $2^n \times 2^n$ where n is the number of qubits involved. Common examples of two-qubit transformations are in the form of the so-called "controlled" operations although other types of multi-qubit operations are also well-known. The controlled two-qubit transformations can be interpreted in an analogous way to electronic gates that utilize an enable or control input such as a tri-state buffer. One qubit serves as the "control" qubit and the other as the "target" qubit. A controlled-**U** operation is an operation wherein the target qubit is transformed according to a 2×2 unitary matrix **U** if, and only if, the control qubit has a non-zero probability amplitude with respect to the $|1\rangle$ basis state. For example, a "controlled-**X**" (or "controlled-*NOT*") gate has a corresponding transfer matrix, $\mathbf{C_X}$, as

$$\mathbf{C_X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \quad (12)$$

C. Representations of Quantum Computations

A quantum computation for the gate model paradigm can be considered as either a dedicated quantum circuit implemented in hardware or as a sequence of instructions to be executed by a QC. For this reason, such computations are commonly referred to as "quantum circuits" although the term "quantum algorithms" would probably be more clear since a "quantum circuit" can represent a sequence of software instructions to be executed on a QC. At present there are several high-level languages emerging for the purpose of writing QC programs as well as representations that would be closer in consideration to a QC assembly language. One such QC assembly language is openQASM [8]. openQASM provides a set of mnemonics for common single- and multi-qubit operations that describe a particular quantum computation wherein each mnemonic represents the transformation of one or more qubits and thus is representative of a transformation matrix such as those shown in Equations (10) and (12). From a software perspective, openQASM can be considered an assembly-level abstraction of QC software and from a hardware implementation point of view, it can be considered as an abstraction analogous to an RTL description.

The QC community uses another common form to describe a quantum circuit that is graphical and can be considered as analogous to an electronic gate-level circuit diagram. Different qubit transformations are represented with symbols and individual qubits supporting the computation are represented as horizontal lines. The convention for these diagrams is that the diagram represents the passage of time from left to right. Thus, a horizontal line representing a qubit indicates the initial quantum state at the leftmost point of the line and it indicates the final state at the rightmost side. Qubit transformations at some point in time are represented by the presence of an operator symbol that intersects one or more qubit lines at the appropriate point in time. As an example, consider the quantum computation shown in Fig. 1. Note that the vertical dashed lines represent specific instances of time and are non-standard annotations added to Fig. 1 to aid in the explanation that follows.

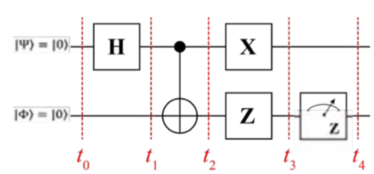

Fig. 1: Example Quantum Computation

In Fig. 1, a two-qubit computation is illustrated where the topmost horizontal line represents qubit $|\Psi\rangle$ and the bottommost line represent qubit $|\Phi\rangle$. On the left side of the diagram the qubits are shown as being initialized to values $|\Psi\rangle = |0\rangle$ and $|\Phi\rangle = |0\rangle$ at time t_0. As the computation proceeds from left to right, $|\Psi\rangle$ undergoes an $\mathbf{H}$ operation at a time following t_0 and thus its state evolves to the value $\mathbf{H}|\Psi\rangle$ at time t_1 while $|\Phi\rangle$ does not have any operation indicated in the interval from time t_0 through time t_1. However, a computation over a collection of qubits should be considered as an operation over the entire quantum state of the system, $|\Psi\Phi\rangle$. Thus, the overall transformation matrix governing the time evolution of the system state from t_0 to t_1 is formed using the outer product, $\mathbf{H} \otimes \mathbf{I}$, and transforms the overall state, $|\Psi\Phi\rangle$ as $(\mathbf{H} \otimes \mathbf{I})|\Psi\Phi\rangle$. This is necessary at least because the system can be in a state wherein the two or more qubits are not mathematically separable; meaning that the overall quantum state vector cannot be expressed as a set of individual two-dimensional vector factors that are combined using the outer product. The identity matrix $\mathbf{I}$ is used since no operator is present to transform $|\Phi\rangle$ before time t_1 whereas $|\Psi\rangle$ undergoes a transformation due to $\mathbf{H}$.

Next, at time t_2, the two-qubit controlled-*NOT* operator is applied to the overall quantum state as indicated by the operator intersecting both horizontal lines immediately preceding time t_2. In this case, the quantum state evolves by multiplying the quantum state at time t_1 with the 4×4 transformation matrix $\mathbf{C_X}$ that represents the controlled-*NOT* gate. After time t_2, two single qubit operations are applied. Qubit $|\Psi\rangle$ evolves according to an $\mathbf{X}$ operation and qubit $|\Phi\rangle$ evolves according to the $\mathbf{Z}$ operator. Once again, these single qubit operators must be applied to the overall quantum state of the system, $|\Psi\Phi\rangle$. The transformation matrix is formed as the outer product, $\mathbf{X} \otimes \mathbf{Z}$ resulting in a state evolution of $(\mathbf{X} \otimes \mathbf{Z})|\Psi\Phi\rangle$ that represents the quantum state at time t_3. A very important convention is that the topmost qubits always correspond to the leftmost operands in calculations of outer product expressions extracted from graphical quantum circuit representations. This convention is important because the outer product operation is not commutative. Finally, immediately preceding time t_4, a measurement operation is applied to qubit $|\Phi\rangle$ and the result of the computation is complete at time t_4.

While this example computation is explained in a step-by-step manner as it progressed, or evolved, in time, it should be noted that the entire computation as a whole can be expressed as a single 4×4 transfer matrix, $\mathbf{T}_{comp}$. In this form, all of the intermediate operations can be combined into a single overall system transfer matrix as

$$\mathbf{T}_{comp} = (\mathbf{X} \otimes \mathbf{Z})\mathbf{C_X}(\mathbf{H} \otimes \mathbf{I}). \quad (13)$$

It is important to form the overall transfer matrix $\mathbf{T}_{comp}$ as the outer product of individual gate transfer matrices in reverse time order to ensure that the operations are applied to the initial system quantum state in the correct time-ordered sequence.

When the overall transfer matrix is formed, the system quantum state at time t_3 can be computed directly using the initial state at time t_0 as

$$|\Psi\Phi\rangle_{t_3} = \mathbf{T}_{comp}|\Psi\Phi\rangle_{t_0}. \quad (13)$$

At time t_4, a projective measurement is made as indicated by the rightmost and lowermost operator applied to qubit $|\Phi\rangle$. The "$\mathbf{Z}$" as annotated on the measurement symbol indicates that the measurement is made with respect to the computational basis and will thus cause the qubit to collapse to either $|0\rangle$ or $|1\rangle$ with the outcome of the measurement being one of the scalar eigenvalues of the Hermitian matrix $\mathbf{Z}$ that models the measurement observable. To determine the probabilities for the possible occurrence of each of these measurement results, we apply Born's rule to the probability amplitudes. At time t_3, the composite quantum state is computed using Equation (13) as

$$|\Psi\Phi\rangle_{t_4} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}. \quad (14)$$

In braket notation, this result is

$$|\Psi\Phi\rangle_{t_4} = \frac{1}{\sqrt{2}}(0 \cdot |00\rangle - |01\rangle + |10\rangle + 0 \cdot |11\rangle). \quad (15)$$

The probability amplitudes for the two basis vectors $|00\rangle$ and $|11\rangle$ are included in Equation (15) to indicate that they are zero-valued. This means that upon observation with respect to the computational basis, only two possibilities can result, either $|01\rangle$ or $|10\rangle$. We observe that the probability amplitude for the $|01\rangle$ basis is $-1/\sqrt{2}$ and that for basis $|10\rangle$ is $+1/\sqrt{2}$. Taking the square of the probability amplitudes yields a probability of 1/2 that $|\Phi\rangle$ will collapse into either $|0\rangle$ or $|1\rangle$. Thus, the measurement is equally likely to cause $|\Phi\rangle$ to collapse into either computational basis state.

It should also be noted that, dependent upon the outcome of the measurement, the topmost qubit $|\Psi\rangle$ must take on the opposite value of $|\Phi\rangle$! This is remarkable since $|\Psi\rangle$ was not measured and was previously in a state of superposition. This is an example of another quantum mechanical phenomenon called "entanglement." Entanglement is a property that can only be created with the application of a multi-qubit operator or "entangling" gate and is responsible for a wide variety of very interesting applications. From a mathematical point of view, entanglement can be observed to occur when the composite quantum state of two or more qubits is such that it is impossible to express the state as an outer product of two single qubit state factors as is the case with $|\Psi\Phi\rangle_{t_3}$ in the example calculation shown in Equation (15).

When two qubits are entangled, the measurement or observation of only one of the host particles can affect the other particle and cause it to lose its state of superposition even when it has not been explicitly measured. This property has profound ramifications and was pointed out by Einstein, *et. al* as a side effect of the theory of quantum mechanics indicative that the theory may be incomplete [9]. However, this question of incompleteness posed by Einstein and his colleagues was later resolved, both theoretically and experimentally, indicating that the principle of quantum entanglement is an accurate and complete consequence of quantum mechanical theory. A recent experiment that currently holds the world record for the longest distance of example of entanglement that was intentionally caused and measured by human activity, over 500 km, is the result of the Chinese government's Micius satellite where they used photon entanglement in a secure encryption key distribution application [10].

The graphical description of the quantum circuit in Fig. 1 can be equivalently specified as an openQASM code fragment as shown in Fig. 2

```
        .
        .
        .
h    psi;
CX   psi, phi;
x    psi;
z    phi;
        .
        .
        .
```

Fig. 2: openQASM Code Fragment for Fig. 1 Circuit

III. Error Sources in Quantum Circuits

The fact that two host particles can become entangled and can exist in this state with very large distances of spatial separation between them means that non-localized correlations can exist among qubits wherein a measurement, whether intentional or not, can affect the state of a computation. While the Micius experiment demonstrated an advantageous use of entanglement, the presence of non-localized spatial correlations among a collection of qubits also indicates the inherent fragility in maintaining a coherent quantum state during a computation.

To improve the reliability of a quantum computation, it is necessary to isolate the implementation from the external environment as much as possible. However, observables at the subatomic scale are microscopically small. Even an extremely minute amount of energy due to heat or cosmic radiation can affect a quantum computation if it happens to interact with a qubit that is in a state of superposition. Such interactions cause information degradation to occur in a computation or even total loss due to decoherence. This is one of the primary reasons that physical qubits are inherently noisy. For this reason (and others) NISQ QC are typically run in a Monte Carlo-like mode wherein a given computation is repeated many times and the output values of the results are accumulated and reported as the result of the computation. The inherent error rates due to environmental interactions and other sources render NISQ QCs as probabilistic computing devices and underscores the dire need for improving QC reliability.

In comparing quantum error models to those for classical electronics, close analogies are that errors due to decoherence are similar to so-called "soft errors" due to intermittent external and natural phenomena such as errors due to radiation. Errors due to device imperfections are closer to so-called "hard-errors" that occur due to manufacturing tolerance variations.

A. Errors due to Decoherence

Errors due to interaction with the environment are considered as decoherence events. The degree to which a QC is coupled to the environment obviously affects the level of decoherence that can be expected. Another factor is the inherent susceptibility of the host-carrying particle to the effects of external influences.

By using models for the stochastic nature of the external environment, two special cases of decoherence can be characterized as "depolarization" and "dephasing." These forms of decoherence are parametrized in the form of decay constants known as the "relaxation time," T_1, and the pure "dephasing time," T_ϕ. When energy levels are used as the information-bearing observable, the relaxation time is related to a decay constant for the amount of time that a qubit will transition or "relax" from a higher energy level to a ground state.' That is, the qubit is said to become "depolarized" when it transitions to the ground state. Typically, the ground state is assumed to represent $|0\rangle$ and the next higher energy level represents $|1\rangle$ when energy levels are used as the information-bearing quantum observable. In this case T_1 defines the average decay time for a qubit to transition from $|1\rangle$ to $|0\rangle$. As an example, the probability that a qubit $|\Psi\rangle$ is in the higher-level energy state $|1\rangle$ at time t after being initialized or evolved to state $|1\rangle$is proportional to

$$P(|\Psi\rangle = |1\rangle) = e^{-\frac{t}{T_1}}. \quad (15)$$

T_ϕ is the "dephasing time" that describes how long the relative phase, ϕ, stays intact. If a qubit is in coherence with a relative phase of ϕ, the dephasing time T_ϕ refers to the time constant of the decay before the relative phase becomes unstable and begins to change or rotate in the *x-y* plane from a geometric interpretation. Typically, $T_1 > T_\phi$, and the time for a QC gate to reliably operate must be less than either of these time constants if a reliable QC is desired wherein a given computation is assumed to consist of hundreds or thousands of single and two-qubit gates.

When depolarization or dephasing occur, they are instantaneous events due to a discrete interaction with the environment as a point process. The exponential decay models can be considered as point process probability mass functions. Furthermore, such events can be considered as uncorrelated, independent, and thus indicative of a Markovian point process.

B. Errors due to Device Imperfections

As previously discussed, a quantum gate operation is specified by a unitary matrix, **U**. When such a gate is implemented in some technology, the physical devices used to implement the operation will be fabricated with some degree of error tolerance. These error tolerances manifest as, and can be represented as, deviations in the form of the ideal gate transfer matrix **U**. Thus, the reliability of any particular instance of unitary transformation gate is related to the distribution of errors present among the tolerance variations. Manufacturing and error models are generally dependent upon the specific form of technology being implemented. If errors due to manufacturing tolerances are modeled as additive then, a general model for the actual operation performed by the manufactured gate is represented as a transformation due to $\mathbf{U}_{actual}$ rather than the intended and ideal transformation **U.** The relationship between **U** and $\mathbf{U}_{actual}$is, $\mathbf{U}_{actual} = \mathbf{U}(\mathbf{I} + \mathbf{U}^\dagger \mathbf{E}_{add})$, where

$$\mathbf{U}_{actual} = \begin{bmatrix} u_{00} + \epsilon_{00} & u_{01} + \epsilon_{01} \\ u_{10} + \epsilon_{10} & u_{11} + \epsilon_{11} \end{bmatrix}, \ \mathbf{E}_{add} = \begin{bmatrix} \epsilon_{00} & \epsilon_{01} \\ \epsilon_{10} & \epsilon_{11} \end{bmatrix}. \quad (16)$$

The coefficients of the $\mathbf{E}_{add}$ matrix can be modeled as random variables according to some appropriate probability density function such as a normal *pdf*.

Error matrices may also be modeled as multiplicative errors and generally this form is more convenient since their effect on a particular computation can be ascertained by simply inserting the error matrix $\mathbf{E}_{mult}$ in the computation of the overall circuit transfer matrix as was demonstrated in Equation (13). The multiplicative error matrix can be inserted into the overall computation by placing it in the chain of direct multiplicative factors in the position corresponding to the time in which the error occurs.

If the manufacturing error manifests as an additive error to the relative phase resulting in an actual phase angle of $\phi + \phi_\epsilon$ rather than the intended ϕ phase angle, then a multiplicative error matrix results indicated as

$$\mathbf{U}_{actual} = \mathbf{U}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi_\epsilon} \end{bmatrix} = \mathbf{U}\mathbf{E}_{\boldsymbol{\phi}}. \qquad (17)$$

Note that the form of $\mathbf{E}_\phi$ is identical to the single-qubit phase gates, **Z**, **S**, and **T**, as given in Equations (10), when the phase error, ϕ_ϵ, takes on values of 180°, 90°, and 45°, respectively.

Numerous variations for formulating $\mathbf{U}_{actual}$ can be derived and are highly dependent on the technology and associated manufacturing processes used. When device imperfection errors are small, they may sometimes be neglected since the measurement process yields a basis state and is thus somewhat self-correcting. For example if a manufacturing error causes a perturbation to the probability amplitudes of a qubit that evolves to $|\Psi\rangle = (\alpha + \alpha_\epsilon)|0\rangle + (\beta + \beta_\epsilon)|1\rangle$ rather than $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$, then for sufficiently small probability amplitude errors, the measurement outcomes are unlikely to change since

$$|\alpha + \alpha_\epsilon|^2 \approx |\alpha|^2, \quad |\beta + \beta_\epsilon|^2 \approx |\beta|^2. \qquad (18)$$

However for large circuits, numerous small errors can accumulate, particularly if the error distributions are not symmetric about a non-zero mean. In this case, manufacturing errors can become significant sources of unreliability.

C. Errors due to Measurement

While measurement can have the desirable effect of self-correcting certain small errors as previously described, it also poses a more serious contribution to QC unreliability due to the entanglement phenomenon discussed in a previous section. When two or more qubits become entangled due to an external environmental effect or a device imperfection, unintended entanglement can cause the measurement of one subset of qubits to cause another set to simultaneously collapse. Because observables are implemented such that they perform a measurement with respect to a particular basis set, known as the measurement basis, measurement device errors can cause the basis set to be imprecise and to deviate from the intended measurement basis. In this case the measurement is with respect to probability amplitudes that are likewise transformed to the measurement basis and can thus affect observed outcomes. This latter type of measurement error is of particular concern in quantum informatics metrology or sensing applications.

IV. Quantum Fault Models

Quantum fault models can be formulated as transformation matrices that are inserted within the intended computation, such as the matrices $\mathbf{E}_{add}$ and $\mathbf{E}_\phi$ in Equations (16) and (17). It is the case that such fault models can be specified in terms of the single-qubit Pauli operators, **X**, **Y**, and **Z**.

Here we consider an error model where some event, whether due to an external disturbance or to a device imperfection, results in an erroneous unitary transformation on one or more qubits, hereafter referred to as a "unitary error." A unitary error event resulting in depolarization can cause a "bit-flipping" error and the corresponding fault model is represented by an unintended Pauli-**X** operation that occurs at the time of depolarization. Likewise, a "phase-flip" error refers to the relative phase changing by 180° and has a corresponding fault model represented by the Pauli-**Z** operator. The Pauli-**Y** matrix represents a phase-flip error followed by a bit-flip error due to the relationship $\mathbf{Y} = i\mathbf{XZ}$. Since the scalar multiple *i* represents a global phase shift of 90° it can be ignored in terms of information content changes.

A single qubit unitary fault model can be formulated as a linear combination of Pauli matrices as

$$\mathbf{E}_{mult} = \epsilon_{\mathbf{I}}\mathbf{I} + \epsilon_{\mathbf{X}}\mathbf{X} + \epsilon_{\mathbf{Y}}\mathbf{Y} + \epsilon_{\mathbf{Z}}\mathbf{Z}. \qquad (19)$$

Born's rule leads to the result that the scalar coefficients of the unitary fault model in Equation (19) must satisfy

$$|\epsilon_{\mathbf{I}}|^2 + |\epsilon_{\mathbf{X}}|^2 + |\epsilon_{\mathbf{Y}}|^2 + |\epsilon_{\mathbf{Z}}|^2 = 1. \qquad (20)$$

Small errors result in a fault model where the $|\epsilon_{\mathbf{I}}|^2$ value is relatively large as compared to the other coefficients.

Multi-qubit fault models can be formulated as outer products of single qubit fault models in the form of Equation (19). A fault model over *k* qubits in an *n* qubit circuit, where $k \leq n$, can be formulated as a linear combination of matrices of dimension $2^k \times 2^k$ with corresponding scalar coefficients. Each individual $2^k \times 2^k$ matrix used to form the sum is itself calculated as the outer product of *k* scaled 2 × 2 Pauli matrices.

Since a qubit can be uniquely specified by a probability amplitude and a relative phase value, the ability to correct phase-flips and bit-flips along with the linearity resulting from the theory of quantum mechanics implies that any arbitrary unitary error has a corresponding fault model that can be specified as a multiplicative error matrix in the form of Equation (19). Furthermore, fault models can likewise be calculated over any number $k \leq n$ qubits using the process described for formulating $2^k \times 2^k$ fault matrices. This observation leads to the encouraging result that the existence of techniques that are fault tolerant based on phase-flip and bit-flip errors are sufficient to correct any possible unitary error.

V. Quantum Error Correction

An extremely active area of current research is focused on quantum error detection and correction (QEDAC). QEDAC is considered to be one of the leading candidates for enabling the evolution to the next generation of QC that utilize fault tolerant logical qubits for computations rather than the direct use of noisy physical qubits in NISQ implementations. While classical error correction methods provide some fundamental principles for QEDAC, the drastic differences in the underlying technology and computational paradigms require new techniques and methodologies to be developed. Classical error correction for fault tolerance has been found to be very robust in terms of fault models. For example, the "stuck-at" fault model has been shown to cover a wide range of different physical faults, even when some of those faults are not, in a strict sense, due to a bit being permanently stuck at a single logic level. Unfortunately, the stuck-at fault model is not particularly useful in QC as are many of the other classical fault models. One reason for this circumstance is that a quantum state is a continuously varying quantity rather than a discrete quantity. Both probability amplitudes and relative phase values are continuous. An analogous fault model that is comparable to classical fault models is the bit-flip model due to depolarization. Somewhat related analogies to QC phase errors may be those related to timing fault models.

Voltage-mode digital electronics enjoys virtually unlimited fanout as long as voltage-restoring buffering is permissible and the accompanying inserted buffer latencies are permissible. A basic premise in classical error correction is the exploitation of redundancy as used, for example, in a repetition code to correct bit-flip errors. For example, a three-bit repetition code can be implemented where three physical bits represent a single logical bit. Such a code is capable of correcting all single bit-flip errors since the logical bit is assumed to take on the value of the majority of physical bits.

Unfortunately, it is theoretically impossible to copy the value of one qubit to that of another, except for the pathological case where the qubit being copied is in a basis state. This impossibility poses a unique challenge for QEDAC and it arises from a fundamental theorem in quantum informatics known as the "no-cloning theorem." The no-cloning theorem places severe restrictions on the use of redundancy in a QC and prevents the direct use of repetition codes that rely on fanout. Nonetheless, there have been significant developments in QEDAC. Although there have been many recent developments in QEDAC, to keep this tutorial succinct and accessible to readers with an introductory knowledge of QC as provided in the previous sections, we will only explore one of the first QEDAC techniques due to Shor [11].

A. Extending the 3-qubit Replication Code into a QEDAC

As an initial step, we will consider the possibility of utilizing a logical qubit, $|\Psi\rangle = \alpha|\overline{0}\rangle + \beta|\overline{1}\rangle$, that is encoded with three physical qubits. We prepare the three qubits to be in a state of superposition with the same probability amplitudes α and β, such that the logical qubit, $|\Psi\rangle$, in terms of the three physical qubits, $|abc\rangle$, is encoded as

$$|\Psi\rangle = \alpha|\overline{0}\rangle + \beta|\overline{1}\rangle \Rightarrow \alpha|000\rangle + \beta|111\rangle. \tag{21}$$

That is, the logical qubit is encoded with the three physical qubits as

$$|\Psi\rangle \Rightarrow |abc\rangle = \alpha|000\rangle + \beta|111\rangle. \tag{22}$$

In such an encoding, single bit-flip errors, indicated by a caret symbol, would cause one of the following representations of physical qubits to occur;

$$|\hat{a}bc\rangle = \alpha|100\rangle + \beta|011\rangle, \tag{23}$$

$$|a\hat{b}c\rangle = \alpha|010\rangle + \beta|101\rangle, \tag{24}$$

$$|ab\hat{c}\rangle = \alpha|001\rangle + \beta|110\rangle. \tag{25}$$

To determine if any of the three physical qubits has suffered a bit-flip error, a two-qubit syndrome state is computed that indicates errors in the physical qubits. Once a single bit-flip error is detected, it can be corrected by conditionally performing another bit-flip in an intentional manner using a $\mathbf{C_X}$ gate. The syndrome computation uses the exclusive-OR or modulo-2 addition operation to compute a pair of observables of the form $|a \oplus c, b \oplus c\rangle$. When measured, the syndrome causes the physical qubits to be projected to basis states due to entanglement relationships between the two syndrome bits and the actual physical qubits. The measured syndrome can take on the following possible values and meaning:

$$|00\rangle \Rightarrow \text{no single bit flip occured}, \tag{26}$$

$$|10\rangle \Rightarrow |a\rangle \text{ single bit flip occured}, \tag{27}$$

$$|01\rangle \Rightarrow |b\rangle \text{ single bit flip occured}, \tag{28}$$

$$|11\rangle \Rightarrow |c\rangle \text{ single bit flip occured}. \tag{29}$$

In the case that a small single-qubit error occurs rather than a bit-flip, one of the probability amplitudes changes by some amount, $\pm\delta$, and the physical bits would have evolved into one of the following six states

$$|\hat{a}bc\rangle = (\alpha \pm \delta)|000\rangle \mp \delta|100\rangle + \beta|111\rangle, \tag{30}$$

$$|\hat{a}bc\rangle = \alpha|000\rangle + (\beta \pm \delta)|111\rangle \mp \delta|011\rangle, \tag{31}$$

$$|a\hat{b}c\rangle = (\alpha \pm \delta)|000\rangle \mp \delta|010\rangle + \beta|111\rangle, \tag{32}$$

$$|a\hat{b}c\rangle = \alpha|000\rangle + (\beta \pm \delta)|111\rangle \mp \delta|101\rangle, \tag{33}$$

$$|ab\hat{c}\rangle = (\alpha \pm \delta)|000\rangle \mp \delta|001\rangle + \beta|111\rangle, \tag{34}$$

$$|ab\hat{c}\rangle = \alpha|000\rangle + (\beta \pm \delta)|111\rangle \mp \delta|110\rangle. \tag{35}$$

As long as the error is sufficiently small, the projective measurement of the syndrome will almost certainly indicate that it has value $|00\rangle$ most of the time with probability $1 - |\delta|^2$ and thus causes the physical qubit to be projected back to its original state utilizing the self-correcting capability inherent in the projective measurement as previously discussed. Even when the unlikely situation that the syndrome measurement does result in a value other than $|00\rangle$, causing the physical qubit with the small error to be projected to its erroneous state, it will be a correct syndrome indicating which physical qubit is in a flipped state and therefore permitting it to be identified and corrected.

The analysis of this extension of the conventional three-bit replication code into the quantum realm by using three physical qubits to represent a single logical qubit has just been shown to correct single bit-flips, including small bit-flip errors. Furthermore, this approach is directly analogous to the conventional three-bit replication code for digital electronics for bit-flip error models. However, as we previously discussed, qubits may also suffer from phase-flip errors and the two-qubit syndrome used here will not detect this class of faults.

A phase-flip error causes the logical qubit in the state $|\Psi\rangle = \alpha|\overline{0}\rangle + \beta|\overline{1}\rangle$ to evolve into the erroneous state $|\Psi_{err}\rangle = \alpha|\overline{0}\rangle - \beta|\overline{1}\rangle$. The likelihood of a phase-flip error occurring is actually increased when the QEDAC just described is employed.

B. The 9-qubit QEDAC due to Shor

One of the first such QEDAC schemes is due to Peter Shor and was published in his 1995 paper [11]. Shor's QEC encodes a single logical qubit into nine physical qubits and can be considered as a further extension of the classical three-bit replication code into quantum technology as was just described for bit-flip errors, but this time it is generalized even more to allow for dealing with phase-flip errors.

The technique requires the preparation of three different groups of three physical qubits resulting in a total of nine different physical qubits that represent a single logical qubit. Each group of three physical qubits is in a form similar to that in the preceding subsection. The idea is that using a replication of three groups of three physical qubits permits comparisons or syndrome-like computations among differences in the various pairs of groups. By comparing pairs of groups of three physical qubits a phase-flip error can be detected. Furthermore, since each group of three qubits is of a form similar to that described in the previous section, single bit-flip errors can also be detected and corrected.

Each logical qubit is representing with the following set of nine physical qubits,

$$|\bar{0}\rangle \Rightarrow \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle), \quad (36)$$

$$|\bar{1}\rangle \Rightarrow \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle), \quad (37)$$

In this encoding, the logical qubits consist of the outer product of each of the three groups of three physical qubits. When a single phase-flip error occurs in one of the groups, an evolution occurs due to a transformation based on a phase-flip fault model transfer matrix. The three-qubit fault model transfer matrix for the phase-flip error is formed as the outer product of the matrix $\mathbf{E}_\phi$ in Equation (17) where $\phi = 180°$, that is $\mathbf{E}_{\phi=\pi} \otimes \mathbf{E}_{\phi=\pi} \otimes \mathbf{E}_{\phi=\pi}$. This phase-flip fault then results in the following state transition to occur in one of the groups of three physical qubits,

$$(|000\rangle + |111\rangle) \rightarrow (|000\rangle - |111\rangle), \quad (38)$$

$$(|000\rangle - |111\rangle) \rightarrow (|000\rangle + |111\rangle). \quad (39)$$

Thus, the fault causes a sign change in one of the three groups of qubits representing the logical qubit. We cannot directly detect the phase-flip fault through application of a projective measurement since that would alter the logical qubit information. The idea is to compare the relative phases of pairs of groups with one another.

To perform the comparisons of the relative phases among a pair of groups, we form an observable over six physical qubits since each pair of groups comprises three physical qubits per group. This observable is formed by flipping the phase of each of the six physical qubits in a given pair. The application of a phase flip twice yields the original phase before the error. Thus the observable will result in measurement values of +1 when the phases are the same within the two groups and a measurement value of -1 when the phases differ. This is due to the theory of projective measurements that indicates the output of a measurement is one of the eigenvalues of the observable, and in fact, it is the eigenvalue corresponding with the measurement basis vector (*i.e.*, the eigenvector since projective observables are Hermitian) that the measurement forces the qubits to collapse into.

The method is then to compute this observable for two sets of groups. For example, the first set can comprise the first six physical qubits and the second set can comprise the first and last three physical qubits. This arrangement of pairs of groups is analogous to the bit-flip syndrome computation in the preceding section except that instead of computing the modulo-2 sum of two physical qubits, we compute and measure the observables formed from phase-flipping fault models. The particular group of three qubits with a phase-flip error can be identified through comparisons of the outcomes of the measurements where we assume the first value is the measurement among the first two groups and the second measurement is among the first and third group.

$$(+1, +1) \Rightarrow \text{no phase flip in the group pair} \quad (40)$$

$$(+1, -1) \Rightarrow \text{phase flip in the third group} \quad (41)$$

$$(-1, +1) \Rightarrow \text{phase flip in the second group} \quad (42)$$

$$(-1, -1) \Rightarrow \text{phase flip in the first group} \quad (43)$$

The outcome of the measurement then conditionally applies an intentional 180° phase-flip, in the form of $\mathbf{Z} \otimes \mathbf{Z} \otimes \mathbf{Z} = \mathbf{Z}^{\otimes 3}$ to the detected group of three qubits that suffered a phase-flip fault. This is accomplished with a controlled-$\mathbf{Z}^{\otimes 3}$ operation where the control is governed by a group that was found to be free of a phase flipping fault. This restores the phase-flip error back to its original state since the direct product $\mathbf{E}_{\phi=\pi}\mathbf{Z} = \mathbf{I}$.

Similar arguments with regard to small errors in phase can be made with this approach as those for the small bit-flip error in the previous subsection. Observables can be formulated that combine the bit-flip syndrome calculation with the phase-flip observable to provide an overall QEDAC that protects against any unitary error since both phase and probability amplitude errors have been addressed, both in terms of the flipping and the small error cases.

C. QEDAC Motivation and Current Efforts

There are numerous new developments in QEDAC; both in terms of newer schema as well as architectural and QC paradigm improvements. We described one of the first QEDAC approaches to introduce the main concepts, yet this pioneering approach due to Shor is now 25 years old. Many current efforts have resulted in very promising approaches and research in quantum error correction is ongoing. A partial list of some of the more current approaches as they were published in the intervening time between the writing of this paper and Shor's original approach are included in the bibliography to provide the reader with a sample of later QEDAC results should they decide to gain a more current understanding of the area. These include [12, 13, 14, 15, 16] and are only intended to represent a sample of the many important advances that have occurred in the field since 1995.

Perhaps the biggest challenge in providing highly reliable quantum informatic processing devices is that of decoherence. Even the noblest and most advanced means of isolating a QC from the external environment is likely to never be entirely sufficient, thus researchers are more focused on determining ways to provide reliable QC assuming that decoherence events will always be present to some degree.

When the quantum state of a QC becomes entangled with "noise" qubits due to external environment interactions before full decoherence occurs, a QEDAC method such as that of Shor also provides some protection since the calculation of the syndrome observables and the associated projective measurements also restore the quantum state of the system back to its original state as a side-effect. That is, the act of computing the syndrome observables and the projective measurements that are involved can be viewed as "dis-entangling" QC physical qubits from previously entangled external environmental quantum states. This observation is responsible for the large amount of activity in the QC community with regard to QEDAC. QEDAC is viewed as one of the most important potential enablers in improving the reliability of QC and can guide the design and implementation of next-generation devices for processing quantum information allowing the availability of much higher-reliability QCs as compared to NISQ devices.

VI. Summary

This tutorial has endeavored to describe the quantum informatic paradigm for processing quantum information with a focus on the differences between the mathematical models describing QC versus those for processing conventional voltage- and current-mode classical models of information. Descriptions of the qubit as an element in the two-dimensional Hilbert vector space are provided and contrasted with the switching theory models used in digital electronics. The concept of information processing, both in the form of software for a QC and as a dedicated quantum informatic hardware circuit are provided in a technology-independent manner. A much more complete reference that provides detailed descriptions of the topics in this paper is [17].

Particular emphasis is placed on reliability of QC processing including a survey of predominant error sources and the development of corresponding fault models in the form of transformation matrices. One of the most challenging error sources, that of decoherence, is explained in a qualitative manner with the aim of providing an intuitional grasp of this important family of error sources. The motivation behind the utilization of the bit-flip and phase-flip fault models as a basis for encompassing a larger group of specific QC fault models is provided. The use of these two fault models as such a basis is somewhat analogous to the use of the "stuck-at" model in conventional electronic processing since the bit-flip and phase-flip models cover a large number of different error sources in a similar fashion as the stuck-at model covers a large variety of error sources in electronic data processing circuits.

The most predominate approach for enhancing reliability, that of utilizing logical qubits in coordination with QEDAC, is developed here by starting with the concept of one of the easiest to understand and simplest forms of conventional error detection and correction; the three-bit replication code. The concept of the three-bit replication code is generalized to encompass quantum information and descriptions of how projective measurements and observables are used to overcome limitations due to the no-cloning theorem are included. The three physical qubit approach for representing a single logical qubit was then generalized to encompass phase errors leading to a description of Shor's nine-qubit QEDAC method [11].

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