Yet another mathematical framework for understanding DSR

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# Motivation

The vast body of literature on Digital Super Resolution fails to provide a definite answer to fundamental questions such as

* Is perfect recovery of an optically band-limited image possible under precise knowledge of translational motion, in the absence of noise?
* What are preferred optical PSF’s and focal plane masks for digital super-resolution?
* Are there super-resolution schemes that are free of regularization?
* What are necessary and sufficient conditions for super-resolution?
* What is the connection between Digital Super Resolution and Papoulis’s Generalized Sampling Theorem?
* What is the connection between Digital Super Resolution and a maximally decimated perfect reconstruction filter-bank?

The present work tries to answer these questions by developing a mathematical framework for understanding Digital Super Resolution.

# Assumptions

1. Circular aperture and space-invariant optical PSF
2. The inter-sample-spacing of the sampled optical PSF given by exceeds the optical cutoff frequency where is the numerical aperture, and is the wavelength of light expressed in meters.
3. An integer number of higher-resolution pixels (size ) can be perfectly accommodated within a single detector pixel (size ). This implies that .
4. The pixels are square in shape, and have 1 fill factor
5. The central pixel in the detector is assumed to be the origin of the image coordinate system.

# Image observation model

|  |  |
| --- | --- |
|  | (1) |

|  |  |
| --- | --- |
|  | geometric image of the scene from the camera vantage point  |
|  | intensity PSF of the optics(accommodates blurring induced by the optics) |
|  | Size of individual pixel in detector |
|  | spatial response of light sensitive element in the detector.(accommodates blurring induced by pixel geometry)  |

****

Figure Relationship between geometric image of scene and detected image

|  |  |
| --- | --- |
|  | optical transfer function |
|  | low resolution detector pixel transfer function  |

# Objective:recover image sensed by detector with smaller pixels

To be precise, we would like to estimate samples from the detected observations . To this extent, it helps to know the expression for

|  |  |
| --- | --- |
|  | (2) |

where

|  |  |
| --- | --- |
|  , s.t  | ideal low-pass filter with the same bandwidth as the optical PSF  |
|  | wavelength in meters |
|  | numerical aperture of the optics |
|  | Desired pixel size |



Figure Relationship between geometric image of scene and desired image

|  |  |
| --- | --- |
|  | high resolution detector pixel transfer function  |
|  | transfer function of rotationally symmetric ideal low pass filter |

**Notes**

The following paragraphs outline a method for recovering samples obtained at intervals of , using a detector with pixel pitch .

# Image observation model (continued…)

With a little effort, (1) can be recast in the following form

|  |  |
| --- | --- |
|  | (3) |

The second expression in (3) follows from the assumption that an integer number of higher-resolution pixels can be perfectly accommodated within a single detector pixel.

Suppose that the optical PSF is band-limited such that its bandwidth does not exceed . In such a case, the 2D extension of the sampling theorem [Pg.88 of A.K. Jain, Oppenheim] may be used to express the optical PSF as follows

|  |  |
| --- | --- |
|  | (4) |

The entries are obtained by directly sampling the optical PSF at intervals of , as shown below

|  |  |
| --- | --- |
|  | (5) |

The constraint that implies that an integer number of higher-resolution pixels (square in shape) can be perfectly accommodated within a single detector pixel (also square in shape). Consequently,

|  |  |
| --- | --- |
|  | (6) |

The weights are defined as follows

|  |  |
| --- | --- |
|  | (7) |

In its present form, (6) suggests that the array has infinite spatial extent. But, in practice it has finite spatial extent and behaves as a Finite-Impulse-Response filter. The spatial response function is expressed as an infinite sum for the sake of mathematical convenience.

Typically, the non-zero components of are identical, corresponding to a summing filter. But, the weights maybe interpreted in a broader context as being derived from a pixelated focal-plane-mask (such as a Hadamard mask) with a feature size of .

Using (4), (6) we will now attempt to evaluate the term in (3).

|  |  |
| --- | --- |
|  | (8) |

Further simplification is possible by utilizing a special property of the convolution and correlation for real signals

|  |  |
| --- | --- |
|  | (9) |

A proof of the above property is included in the Appendix. In view of the property, one may simplify (10) as follows

|  |  |
| --- | --- |
|  | (10) |

Additional simplification is possible by utilizing a It follows from the definition of the Dirac-delta function that

|  |  |
| --- | --- |
|  | (11) |

In view of the above property, one may simplify (10) as shown below

|  |  |  |
| --- | --- | --- |
|  |  | (12) |

Under the change of variables, the left operand of the convolution may be expressed as the cross-correlation of the discrete arrays, so that

|  |  |
| --- | --- |
|  | (13) |

The term may be interpreted as the expression for a band-limited image that is reconstructed from uniformly spaced samples with an inter-sample-spacing of . In subsequent expressions, the aforementioned term will be denoted as the effective PSF.

Substituting (13) in (1) yields the following expression for the intensity of the pixel of the detected image

|  |  |
| --- | --- |
|  | (14) |

Diffraction limits the largest spatial frequency that can be resolved by the optics to , where is the wavelength of illumination, and is the numerical aperture.

Consequently, post-factum filtering of by an ideal low-pass filter such as has no effect on. In other words, . In view of this observation, one can rewrite (14) as follows

|  |  |
| --- | --- |
|  | (15) |

The definition of the term in (2) is utilized to derive the final expression in (15).

 According to the Shannon-Whittaker sampling theorem, the band-limited image admits the following expansion

|  |  |
| --- | --- |
|  | (16) |

We know from (13) that the effective PSF admits a similar series expansion. By combining the identities (13) & (16), one can determine the expression for the image, which when sampled at regular intervals yields the intensity of the detector pixel. The result is included below

|  |  |
| --- | --- |
|  | (17) |

In deriving (17), the property was utilized.

With the aid of (17), one can deduce the intensity of the detector pixel. The result is included below

|  |  |
| --- | --- |
|  | (18) |

In deriving (18), the following property of the function is utilized

|  |  |
| --- | --- |
|  | (19) |

In summary, the intensity of the pixel may be obtained as follows

1. uniformly sample the band-limited image at intervals of width , to obtain the image
2. convolve the sampled image with the array where is the sampled optical PSF, and is the sampled detector spatial response
3. down-sample the result by a factor of

****



Figure Relationship between the geometric image of scene and the desired, detected images

|  |  |
| --- | --- |
|  | Discrete Time Fourier Transform of the sampled optical PSF  |
|  | Discrete Time Fourier Transform of Facts* In the absence of a focal plane mask

.* The array has exactly non-zero entries when is an integer. These entries are in the absence of a focal plane mask, and correspond to a code in the presence of a focal plane mask
 |

# Influence of “detector” shifts on the detected image

For the purpose of this analysis, we assume that the detector experiences a translational motion of meters, prior to integration. It will be assumed that

By definition,

|  |  |
| --- | --- |
|  | (20) |

In an effort to evaluate (20) we recall the expression for listed in (13),

|  |  |
| --- | --- |
|  | (21) |

The definition of the various terms referenced in (21) is included below

|  |  |
| --- | --- |
|  | (22) |

We know from the discussion in the previous section that the effective PSF is unaffected by convolution with an ideal low pass filter, i.e.

|  |  |
| --- | --- |
|  | (23) |

Incorporating (21), (22) and (23) into (20) yields the following expression for the intensity

|  |  |
| --- | --- |
|  where | (24) |

Following (17), we can express as follows

|  |  |
| --- | --- |
|  | (25) |

Incorporating (25) into (20) yields the following expression for the intensity

|  |  |
| --- | --- |
|  | (26) |

In deriving (25), the following properties of the function is utilized

|  |  |
| --- | --- |
|  | (27) |

In summary, the intensity of the low-res image pixel for a detector translation of meters, may be computed as follows

1. uniformly sample the band-limited image at intervals of width , to obtain the image
2. convolve the sampled image with the array where is the sampled optical PSF, and is the sampled detector spatial response
3. down-sample the result by a factor of

# Influence of “object” shifts on the detected image

For the purpose of this analysis, we assume that the object experiences translational motion of meters, prior to imaging. It will be assumed that

By definition,

|  |  |
| --- | --- |
|  | (28) |

In an effort to evaluate (20) we recall the expression for listed in (13),

|  |  |
| --- | --- |
|  | (29) |

The definition of the various terms referenced in (21) is included below

|  |  |
| --- | --- |
|  | (30) |

We know from the discussion in the previous section that the effective PSF is unaffected by convolution with an ideal low pass filter, i.e.

|  |  |
| --- | --- |
|  | (31) |

Incorporating (21), (22) and (23) into (20) yields the following expression for the intensity

|  |  |
| --- | --- |
|  where | (32) |

Following (17), we can express as follows

|  |  |
| --- | --- |
|  | (33) |

Incorporating (25) into (20) yields the following expression for the intensity

|  |  |
| --- | --- |
|  | (34) |

In deriving (34), the following properties of the function is utilized

|  |  |
| --- | --- |
|  | (35) |

In summary, the intensity of the low-res image pixel for a detector translation of meters, may be computed as follows

1. uniformly sample the band-limited image at intervals of width , to obtain the image
2. convolve the sampled image with the array where is the sampled optical PSF, and is the sampled detector spatial response
3. down-sample the result by a factor of

# Exact recovery of the optical image in noiseless DSR: Fact or fiction?

Optimization problem we wish to solve ??

Need to work on this section

Key points

* One can reconstruct in the absence of noise only iff
	+ There are no nulls in the DTFT of the sampled optical PSF
	+ There are no nulls in the DTFT of the sampled detector response

In mathematical terms, this implies that and for .

Failure to meet these constraints result in an irreversible loss of the frequencies. This suggests that clear pixels with large footprint may be not be well suited for perfect recovery of the optical image.

Example

Suppose. Consider a two-valued high-res spatial pattern that has intensities in columns and intensities in columns .The frequency of this pattern is

Sub-pixel shifting followed by integration over neighborhood always yields the same value, so that is always gray. No amount of regularization can help recover this frequency. The most appropriate prior for reconstruction is a sparsity prior for the object, but even that will not work as a constant gray image is the sparsest solution to the regularization problem.

Question

Is there a non-uniform spatial pattern such that for all. If so, will any reconstruction algorithm claim that is constant over a neighborhood? I think the answer is yes, and these patterns correspond to nulls in.

* Identify necessary and sufficient condition for exact recovery as

# Derivation of the 1D Generalized Sampling Theorem (GST)

**Claim**: *A band-limited function is uniquely determined in terms of samples of the responses of linear systems with input, sampled at of the Nyquist rate.*



Figure Generalized sampling configuration

**Comments**

The GST of Papoulis deals with the configuration shown in **Figure 4**, wherein a single band-limited signal is fed into linear time-invariant filters (pre-filters), and each filter output is then sampled at the Nyquist rate. Under certain conditions on the pre-filters and in the absence of noise, the input can be reconstructed “exactly” from the samples acquired by sampling the filter outputs.

**Notes**

The following derivation borrows heavily and shamelessly from the following sources:

1. “Generalized sampling expansion” by A. Papoulis
2. “Multichannel sampling of low-pass signals” by John L. Brown Jr.
3. “Classical sampling theorems in the context of multirate and polyphase digital filter bank structures” by P.P. Vaidyanathan and V.C. Liu

**Proof**

For the purpose of this proof, it is assumed that the signal is band-limited and square integrable, such that its spectrum vanishes outside the interval, where is the sampling inerval. In subsequent discussions, the sampling rate will be denoted by the symbol

The proof begins by identifying the expression for the sampled output of the filter in the bank of pre-filters,

|  |  |
| --- | --- |
|  | (36) |

The spectrum of the sampled signal is given by

|  |  |
| --- | --- |
|  | (37) |

Following a change of variables, (37) may be rewritten as shown below

|  |  |
| --- | --- |
|  | (38) |

The sampled signals serve as input to the reconstruction filters. The expression for the reconstructed signal   is given below

|  |  |
| --- | --- |
|  | (39) |

The spectrum of the reconstructed signal is given by

|  |  |
| --- | --- |
|  | (40) |

Perfect recovery of the signal demands that. But then, (40) in its present form provides little insight into the existence of analysis and synthesis filters such that.

In an attempt to address the issue, we gather terms in the outer summation of (40) whose indices yield a reminder of when divided by, i.e.

|  |  |
| --- | --- |
|  | (41) |

The expression for provided in (41) may be simplified by utilizing the one-to-one correspondence between elements of the set and the set .

The simplified expression for   is shown below

|  |  |
| --- | --- |
|  | (42) |

Additional simplification is possible by recognizing that the passband of the reconstruction filters need only be confined to the interval, given that the spectrum of the input signal vanishes outside the interval.

Please note that the above restriction is in line with the established practice of reconstructing a band-limited signal from its un-aliased samples, by employing an ideal LPF whose cutoff frequency matches the Nyquist frequency.

Incorporating the above constraint in the expression for amounts to discarding terms that correspond to. The final expression is shown below

|  |  |
| --- | --- |
|  | (43) |

Careful inspection of (43) yields a set of constraints must be simultaneously satisfied for perfect recovery, via. These include

|  |  |
| --- | --- |
|  | (44) |

The above constraints may be recast as the following matrix identity

|  |  |
| --- | --- |
|  | (45) |

Please note that the above constraints are strictly valid for, as vanishes outside this interval. This requirement may be incorporated into (45), to yield the following revised constraints

|  |  |
| --- | --- |
|  | (46) |

It is obvious from (45) that the transfer function of the post-filters namely , are specified by the first column of scaled by.

Please note that

* (45) is identical to Eq-11 in the paper “Multichannel sampling of low-pass signals” by when , and under the following substitutions

, , ,

Also, the notion that the transfer function of the post- filters are specified by the first column of is consistent with Eq-12 of the paper.

* (45) is identical to Eqs-(41,42) listed in the book chapter “Superresolution Imaging - Revisited” by Markus E. Testorf and Michael A. Fiddy, under the following substitutions

, ,

* (45) is identical to Eq-(4-21) listed in the book “Systems and Transforms with Applications in Optics” by A. Papoulis, under the following substitutions

,, ,

 and

 PLEASE VERIFY THE LAST ENTRY FORMALLY!!!

Intuitive meaning of (46)

According to the GST, perfect reconstruction of demands that (46) be satisfied, requiring that each column of the matrix is different from a column of zeros. This is only possible when the aggregate bandwidth of the filters spans the interval. In other words, every frequency within the interval must be admitted by one of more of the filters .

TODO: Tie above argument with discussion on well-posed GST in “On well-posedness of the Papoulis Generalized Sampling Expansion”…specifically Theorem-1.

# Connection between GST and Digital Super Resolution



Figure Interpreting Digital Super Resolution as a variant of the standard GST configuration

|  |  |
| --- | --- |
|  | Size of individual pixel in detector |
|  | Desired pixel size  |
|  | Point Spread Function |
|  | Optical Transfer Function Note: The OTF is a band-limited transfer function on account of diffraction |
|  | Detector spatial response (square pixels with 100% fill factor) |
|  | Detector Transfer Function( accommodate blurring induced by gathering light over the finite area of a single pixel )  |
|  | Input to the bank of pre-filters |
|  | Transfer function of pre-filter ( due to sub-pixel shifting ) |
|  | Transfer function of post-filter( Ideal low pass filter + linear phase )  |

The following choice of pre-filters and post-filters guarantee perfect recovery of the image, when using a detector with a pixel pitch of

|  |  |
| --- | --- |
|  | (47) |

The transfer function of the pre-filters correspond to an ideal delay line, while the transfer function of the post-filters correspond to delayed ideal low-pass filters.

It can be readily verified that for

|  |  |
| --- | --- |
|  | (48) |

Likewise,

|  |  |
| --- | --- |
|  | (49) |

Note: The shifts in the definition of the pre/post-filters need not be integer multiples of . They could be arbitrary shifts, so long as the pre/post-filters are perfectly matched.

Expression for the subsampled image in the channel

Expression for the reconstructed image

Details

* It should be evident from **Figure 5** that one can only reconstruct the band-limited image in the absence of noise. This implies that any frequencies lost to optical blurring may never be recovered. Likewise, nulls in result in an irreversible loss of the null frequencies. This suggests that clear pixels with large footprint may be not be well suited for perfect recovery of the optical image.
* So long as is devoid of nulls in the interval , one can recover using an inverse filter.

If one wishes to super-resolve by, one could choose

The resulting transfer function does not have any nulls in the interval .

It is worth examining if this idea can be generalized further.

# Digital Super Resolution - Reduction to practice

In practice,

* The sampled optical PSF has finite spatial extent, say pixels. Likewise, the spatial response of a single detector pixel spans higher-resolution pixels. This implies that the cross-correlation has a spatial extent of pixels. In conclusion, the intensity of the detector pixel may be inferred from no more than independently weighted observations of.
* The detector has finite area.

# TODO

* Give the finite spatial support of the optical PSF & the detector integration mask Rewrite the expression for using matrix algebra,
* Say we are interested in recovering the image up-to optical blur so that the following expression is true

This implies that and.

, since the optical PSF has been absorbed into the definition of . In such a case, one finds that

Likewise,

The above expression suggests that the intensity of a single low-res pixel is a simple linear combination of high-res pixel intensities.

# Other observations

We know from (18) that . In the special case that, it follows that

|  |  |
| --- | --- |
|  | (50) |

It is obvious from (50) that the intensity of the detector pixel is identical to the intensity of the sample of the band-limited image.

# Lingering Questions

* How does space-variance in the pixel transfer function of each detector pixel affect the expression for ?
* What happens when the sub-pixel shifts are not integer multiples of ?
* What if I want to super-resolve by a factor ? Does our analysis support this?
* What if I wanted to super-resolve by a rational number?

# Appendix – Useful identities

1. For real signals

|  |  |
| --- | --- |
|  | (51) |

1. For integer arguments

|  |  |
| --- | --- |
|  | (52) |

The above property of the sinc function makes it a cardinal function, which are functions that take the value 0 on the non-zero integers, and take the value 1 at 0.

1. Cross-correlation identities for real signals

|  |  |
| --- | --- |
|  | (53) |

|  |  |
| --- | --- |
|  | (54) |

|  |  |
| --- | --- |
|  | (55) |

# Accommodating fill factors

Replace with.

|  |  |
| --- | --- |
|  | pixel size in detector |
|  | desired pixel size |
|  | Response of the pixel in the higher-resolution detector. |
|  | Response of the detector pixel. |
|  | Response of the detector pixel. |

Update definition of

|  |  |
| --- | --- |
|  | (56) |