Queuing aspects of multiantenna multiple access channels

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***Abstract*— In this communication, we investigate the effect of spatial dimension in a Gaussian multiple access system and study its impact on the physical layer capacity and the end to end delay from a queueing theory perspective. For a multiple-input, single-output (MISO) multiple access channel, we derive conditions on the transmit power and number of transmit antennas for which additional users increase or decrease the capacity. The capacity as a function of the number of users is used for calculating the end-to-end delay. Numerical analysis on the end-to-end delay for a Rayleigh fading MISO channel with no channel state information at transmitter (CSIT) indicates that increasing the number of transmit antennas can actually increase the delay!**

Index terms– MIMO, MAC, Queueing theory

I. Introduction

The study of information and queueing theories have developed along largely independent paths. The importance of creating a unified framework for studying channel variations and packet arrivals in a communication system has been well recognized [1], [2], [3]. A simplified model for Gaussian multiple access channel is proposed in [4] to study the effect of packet delay and physical layer design. More recently, cross layer optimization techniques are receiving increasing attention from many researchers; *e.g.* see [5], [6], [7] and references therein.

In this paper, we study the interaction between random packet arrivals and a fading multiple access channel (MAC). Specific focus is on understanding the impact of multiple transmit/receive antennas on the queueing performance of the incoming traffic. As a side result, we also derive the capacity for a multiple antenna, multiple access channel assuming single user decoding (SUD) at the receiver. This paper also characterizes the limiting behavior of the capacity in certain special cases. Bounds on the achievable delay are derived for various configurations of physical layer including power, probability of error and number of transmit and receive antennas.

The main results of this paper can be summarized as follows: i) For a time-invariant (non-fading) multiple input, single output (MISO) MAC, with identical received power for each user, the sum-capacity is independent of the number of transmit antennas at each user. For instance, some users could have a single transmit antenna and other users could have large number of transmit antennas: As long as each user has constant received power, the throughput is a constant. ii) For a system with large number of users, the sum-capacity under single user decoding approaches a constant for all values of transmit power (The sum-capacity is also constant for any number of transmit antennas, if received power is constant as stated in point (i).), iii) The variation of sum-capacity with number of transmit antennas, for a MISO MAC fading channel indicates an unexpected (and non-intutive) behavior. For a single user scenario, it is well known that adding more transmit antennas increases the capacity [8]. However, for a MAC with SUD, the capacity could decrease with increasing number of transmit antennas and iv) The end-to-end delay reduces with increasing number of receive antennas for a fixed number of transmit antennas and increases with increasing number of transmit antennas for a fixed value of number of receive antennas.

We assume that channel state information (CSI) is not available at the transmitter but perfect CSI is available at the receiver. We consider both a fixed channel and a block fading channel. We consider the noise at the receiver and the fading gains as uncorrelated complex Gaussian processes. To stay in focus, we model the arrival traffic as a Poisson process. In this paper, we only consider single user decoding at the receiver; SUD is motivated by three things: i) the receiver complexity is low, ii) closed form analysis is feasible in most cases and iii) the random arrival of messages adds a new dimension of complexity to sophisticated joint multiuser decoders.

The remainder of this paper is organized as follows: Section 2 introduces the system model, extends a key result in multiantenna theory to multiple access channel MIMO system and briefly review queueing theory for multiple access channels. In Section 3 we derive results for the MISO multiple access channel with all users seeing identical channels and for constant receive power. Section 4 studies the delay for a Rayleigh fading MIMO multiple access channel. We conclude with interesting insights learned in Section 5.

II. System Model and Preliminary Analysis

Consider a Gaussian multiple access channel (MAC) with multiple transmit and receive antennas (MIMO). In the multiple access scenario, multiple users communicate to a single receiver. For the MIMO MAC, each user has N_t transmit antennas while the receiver has N_r receive antennas. There are $u(t)$ users in this system at time t, each of whom is constrained to have a total power of P Watts. The transmitter has no knowledge of the channel and hence the transmit power

P for each user is distributed equally among all the transmit antennas. Messages arrive at the transmitter according to a Poisson distribution with arrival rate λ. In order to avoid considering message queues at the transmitter, we assume that each message is handled by one user. Thus, $u(t)$ represents the total number of messages of all users.

If each message is of length $\log_2 M$ bits, there is a total of M possible codewords, one of which is transmitted by the encoder of a particular user. The i^{th} user encodes its message into a random infinite length codeword, where each element of the codeword is an N_t dimensional transmitted vector $\mathbf{x} \in \mathbb{C}^{N_t}$. Although, the codeword is of infinite length, following [4], we assume that there exists a mechanism for the transmitter to transmit only a finite length of the codeword while ensuring asymptotically *error free* decoding at the receiver. For instance, the receiver could send a feedback to the i^{th} transmitter (user) indicating successful decoding of that message; the corresponding transmitter could then cease transmission of the codeword. We consider single user decoding for each user and hence the decoder treats the signal from $u(t)-1$ other users as interference for a given user. The N_r dimensional received signal $\mathbf{y}$ is given by,

$$\begin{aligned} \mathbf{y} &= \sum_{k=1}^{u(t)} \mathbf{H_k x_k} + \mathbf{z} \\ &= \mathbf{H_i x_i} + \sum_{j=1, j\neq i}^{u(t)} \mathbf{H}_j \mathbf{x}_j + \mathbf{z}, \end{aligned} \tag{1}$$

where, $\mathbf{x_i} \in \mathbb{C}^{N_t}$ is the transmit signal of the i^{th} user, $\mathbf{H_i} \in \mathbb{C}^{N_r \times N_t}$ is the channel matrix for user i and $\mathbf{z} \in \mathbb{C}^{Nr}$ is a complex Gaussian noise at the receiver with 0 mean, independent, equal variance, real and imaginary parts. The noise vector $\mathbf{z}$ has power spectral density $N_o/2$, two sided bandwidth W and covariance $\mathcal{E}\left[\mathbf{z}\mathbf{z}^\dagger\right] = K_{N_r}$. For simplicity, we select $K_{N_r} = N_0 W \mathbf{I}_{N_r}$. The transmit power constraints imply $\mathcal{E}\left[\mathbf{x_i}^\dagger \mathbf{x_i}\right] \leq P$. In (1), we have split terms to indicate the desired signal for the i^{th} user and the interference from other users.

We consider two different models for the channel matrix namely deterministic channel and random channel and denote them as $\mathcal{H}$ and $\mathbf{H}_i$ (for the i^{th} user) respectively. We consider only a special case of the deterministic channel, in which the effective channel gain is identical for all users, *i.e.*,$\|H_i\|_F = \|H_j\|_F, \forall i,j$, where $\|.\|_F$ is the Frobenius norm. This choice corresponds to having equal receiver power for all users since the transmit power constraint is the same for all users. For the random channel, we assume $\mathbf{H}_i$ to have complex Gaussian entries. Both the real and imaginary parts are independent with mean 0 and variance 0.5.

In [4], the authors use the random coding bound on the probability of error to draw a parallel between the random coding exponent term E_o and the service rate for the multiple access queue. The random coding bound is given by [9],

$$P_e \leq \exp\left[\rho \ln M - \sum_{i=1}^{u(t)} E_o\left(\rho, \mathbf{H_i}, q_x\right)\right] \tag{2}$$

where, $E_o\left(\rho, \mathbf{H}, q_x\right)$ is given by the supremum of (3) over all input distributions q_x of the codeword. In general, the q_x that maximizes the error exponent is given by a distribution concentrated on a "thin spherical shell" [8]. However, choosing q_x as the Gaussian distribution lends itself useful for simplified expressions and a convenient lower bound on the E_o(and consequently an upper bound on the probability of error). Therefore, we consider the distribution of $\mathbf{x}$ to be zero mean Gaussian with covariance $\mathbf{Q} = \mathcal{E}\left[\mathbf{x}\mathbf{x}^\dagger\right]$. Since, we assume that channel knowledge is not available at the transmitter, $\mathbf{Q} = \frac{P}{N_t}\mathbf{I}_{N_t}$. In [8], the error exponent for the single user case has been derived. For the multiple access case, E_o can be derived as follows. Without loss in generality, we compute $E_o\left(\rho, \mathbf{H_1}, \mathbf{Q}\right)$ for user 1. From definition of error exponent for continuous channels [9],

$$E_o\left(\rho, \mathbf{H_1}, \mathbf{Q}\right) = -\ln \int \left[\int q_x p\left(y|x\right)^{1/(1+\rho)} dx\right]^{1+\rho} dy \tag{3}$$

where,

$$\begin{aligned} p\left(y|x\right) &= \det\left(\pi K\right)^{-1} \exp\left(-\left(y - \mathbf{H_1 x}\right)^\dagger \mathbf{K}\left(y - \mathbf{H_1 x}\right)\right) \\ \text{and } K &= \mathcal{E}\left[\mathbf{z}\mathbf{z}^\dagger\right] + \sum_{k=2}^{u(t)}\sum_{l=2}^{u(t)} \mathbf{H_k}\mathcal{E}\left[\mathbf{x_k}\mathbf{x_l}^\dagger\right]\mathbf{H_l}^\dagger \\ &= K_z + \sum_{k=2}^{u(t)} \mathbf{H}_k \mathbf{Q} \mathbf{H}_k^\dagger \end{aligned} \tag{4}$$

K_z and K are $N_r \times N_r$ matrices. By substituting the expression for the conditional density in (3) and evaluating the integral, it can be easily shown that the error exponent is given by,

$$E_o = \rho \ln\left[\frac{\det\left(K + \frac{\mathbf{H_1 Q H_1^\dagger}}{1+\rho}\right)}{\det K}\right], \tag{5}$$

where the free parameter ρ can be numerically selected to minimize delay [4] or minimize the probability of error [9]. For the single user case, the error exponent in (5) coalesces to that in [8]. An interesting link between the physical layer and the network layer was brought out in [4] by relating the capacity to the service rate and the specification on a desired probability of error and message size to the demand. A numerical evaluation of the end-to-end delay was obtained by considering a processor sharing model as given in [10]. In this communication, we focus on the impact of spatial diversity on the end-to-end delay and hence summarize the formulation in terms of a processor sharing model and the calculation of the average delay from [4] and [10].

If each user who tries to communicate to a receiver can be thought of as a job in a processor sharing system, then we have a demand

$$S = -\log P_e + \rho \log M.$$

The service rate at time t is $\phi(u) = Wu(t)E_o(\rho, \mathbf{Q}, \mathbf{H})$. The average delay in servicing these users(or the average transmission duration) can be obtained by Little's Law [11] which relates the delay to the average number of users in the system and the arrival rate of the users. Thus, the average delay is given by,

$$\bar{D} = E[U]/\lambda, \tag{6}$$

where, U, the number of users in the system has a distribution given by,

$$Pr\{\text{u jobs in the system}\} = \frac{1}{C\phi_!(u)}(\lambda\mathcal{E}[S])^u \tag{7}$$

with,

$$\phi_!(u) = \prod_{\nu=1}^{u(t)} \phi(\nu) \text{ and } C = 1 + \sum_{u=1}^{\infty} (\lambda\mathcal{E}[S])^u / \phi_!(u) \tag{8}$$

A condition for stability of the system in this case can be derived in a manner similar to [4] and is not shown here. A numerical evaluation of the delay v/s probability of error can therefore be obtained if the capacity is known as a function of the number of users in the multiple access channel.

III. Effect of Multi User Interference on MISO Capacity

In this section, we study the capacity of the MISO multiple access channel as a function of the number of users. Particularly, we study the variation of the capacity with the number of users for certain multiantenna channels. Let us consider the simple case of identical effective channel gains $\mathcal{H}$ for all the users. If we fix $N_r = 1$ and vary the number of transmitting antennas, the sum rate capacity using single user decoding can be easily derived as,

$$C = uW\ln\left(1 + \frac{\beta}{N_oW + \beta(u-1)}\right) \tag{9}$$

where,

$$\beta = \frac{P}{N_t}\sum_{i=1}^{N_t} |\mathcal{H}_{1i}|^2$$

The capacity thus depends upon the number of transmitters, the allocated power and the number of users. Let us consider two scenarios: small and large β: Large and small β can be thought of as high and low SNR for a fixed N_o.

A. Large β

For $u = 1$, the capacity reduces to $W\ln\left(1 + \frac{\beta}{N_oW}\right)$. For large u it turns out that $(u-1)\beta >> N_oW$, and using the fact that $\ln\left(1 + \frac{\beta}{N_oW+\beta(u-1)}\right) \approx \frac{\beta}{N_oW+\beta(u-1)}$ the capacity approaches W. Since β is large, $\ln(1+\frac{\beta}{N_oW}) > 1$ and it *seems like* the capacity decreases as the number of users increases. However, the variation of capacity with u is not monotonic and is characterized by Lemmas 1 and 2.

B. $\beta << N_oW$

For $u = 1$, since $\beta << N_oW$, the capacity approaches β/N_o (since $\log(1+x) \to x$ for small x). For large number of users and fixed β, the u term dominates similar to the large β case and hence the capacity approaches the same limit of W. Thus, it *seems like* the capacity increases from β/N_o to W as the number of users increases. However, as noted before, the variation of capacity with u is not monotonic as characterized by Lemmas 1 and 2.

This contrasting behavior of the capacity at low and high SNR is plotted in Fig. [1] for the deterministic MISO MAC. As noted before, for large u, the capacity $C \to W\forall\beta$. However, it is interesting to see that for any given u^*, the capacity can equal W for an appropriate choice of β^*. This β^* is obtained by equating the right hand side of (9) to W. Consequently, β^* can be derived as,

$$\beta^* = \frac{N_oW}{1 - u + \frac{1}{e^{1/u}-1}}. \tag{10}$$

Now, for this choice of β^*, the capacity equals W both at $u = u^*$ and at $u \to \infty$; thereby conforming the non-monotonic behavior in Fig. [1]. The figure shows the variation of capacity for $\beta^* = 1.71, 1.84, 1.89$ corresponding to $u^* = 1, 2, 3$, respectively. In the following two Lemmas, we quantify the slope of the capacity v/s users profile.

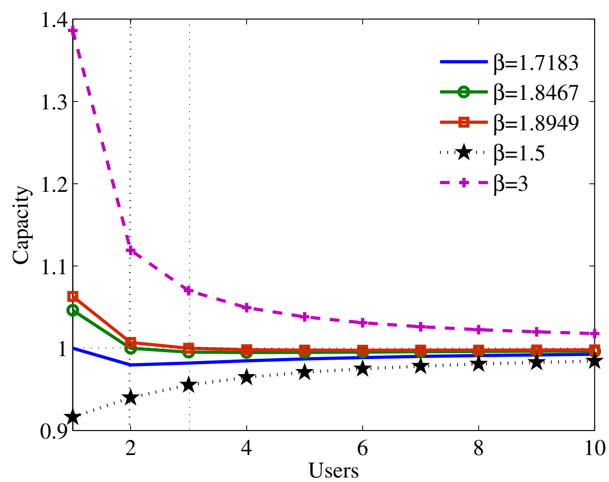

Fig. 1. Plot of Capacity v/s Number of users for a deterministic MISO Multiple Access Channel. For the choice of $\beta = 1.71, 1.84, 1.89$, Capacity equals W when the number of users is 1, 2 and 3 respectively and as $u \to \infty$. For $\beta = 1.5, 3$, the Capacity equals W only as the number of users increases.

C. Characterization of the Capacity v/s number of users curve

Lemma 1 : For a Gaussian MAC with single receive antenna and identical channels for all users, the slope of the capacity v/s number of users is positive at u=1 if

$$\ln\left(1 + \frac{\beta}{N_oW}\right) > \frac{\beta^2}{N_oW(N_oW + \beta)} \tag{11}$$

and negative otherwise. Further, if $\beta < N_oW$, the slope will be positive at $u = 1$.

Proof : For the slope of the Capacity versus number of users to increase w.r.t u at $u = 1$, the slope should be positive. By differentiating the sum rate Capacity w.r.t u, we get

$$\frac{\partial C}{\partial u} = \ln\left(1 + \frac{\beta}{N_oW + \beta(u-1)}\right) - \frac{u\beta^2}{(N_oW + \beta(u-1))(N_oW + \beta u)} \quad (12)$$

For the slope to be positive at $u = 1$, if we substitute $u = 1$ in (12) and apply conditions for the expression to be positive, we obtain (11). Further, for small β, by expanding the logarithmic term in its Taylor series, retaining only the first 2 terms of the expansion and by rearranging terms, we get,

$$\beta^2 + \beta N_oW - 2N_o^2W^2 < 0 \quad (13)$$

for the slope to be positive. Solving the quadratic equation in β, we get $\beta < N_oW$ ensures that for $\beta/N_oW << 1$, the slope will be positive. □

Similar conditions can be obtained for the slope to be positive or negative at any desired u. The important implication of lemma 1 is that at u=1, the curve of capacity v/s number of users can either be increasing or decreasing with u. The following lemma characterizes the slope of the curve for large u.

*Lemma 2:*For a Gaussian MAC with single receive antenna and identical channels for all users, in the limit of large number of users, the slope of the capacity versus the number of users curve is always positive.

Proof: Similar to the proof in Lemma 1, we differentiate (9) w.r.t u. For large u, the argument of the log term in the derivative in (12) approaches $\beta/\left[N_oW + \beta(u-1)\right]$. The second term approaches $\frac{\beta}{\beta u + 2N_oW}$. By rearranging terms, we see that for large u,

$$\frac{\partial C}{\partial u} = \frac{\beta\left(N_oW + \beta\right)}{\left(N_oW + \beta u - \beta\right)\left(2N_oW + \beta u\right)} \quad (14)$$

which is always positive for large u and any $\beta > 0$. Further, as $u \longrightarrow \infty$, the slope approaches 0 indicating that the capacity converges. □

Lemma 2 has an interesting implication: Since the slope of the capacity v/s u is positive for large u, for the choice of transmit power and N_t that satisfy $\beta > N_oW\left[\sqrt{5} - 1\right]/2$, although the capacity decreases with the number of users for small u, it goes below the threshold W for some u and starts increasing thereafter. Addition of users in the multiple access MISO system can therefore increase or decrease capacity depending upon the choice of N_t and the SNR.

D. Varying N_t

For the same single receiver antenna case with identical channel gains, we now consider the impact of the additional degree of freedom of the number of transmitters, N_t. If $\mathcal{H}$ and $\mathcal{H}^*$ are the channels for N_{t1} and N_{t2} respectively, a sufficient condition for the channels to have the same capacity is obtained by equating the capacities in (9) as:

$$\frac{P}{N_{t1}}\sum_{i=1}^{N_{t1}}|h_{1i}|^2 = \frac{P}{N_{t2}}\sum_{j=1}^{N_{t2}}|h_{1j}^*|^2 \quad (15)$$

Effectively, this relation tells us that if the received power is the same for two different number of antennas at the transmitter, the capacity is the same.

For the single user Rayleigh fading channel, with a fixed number of receive antennas, the capacity approaches $N_r \log(1 + P)$ for large N_t [8]. In the MISO case, for random $\mathbf{H}$, numerical analysis indicate that for more than 1 user, the capacity is a non-increasing function of the number of transmit antennas for any value of transmit power as shown in Fig. [2]. The figure shows that the capacity can increase or decrease with the number of users in the case of random channels similar to deterministic case. However, the rate of change is different for various values of N_t. Consequently, for certain values of u, a particular N_{t1} could give rise to higher capacity than N_{t2} but for a different u, the capacity with N_{t2} antennas could be higher than with N_{t1} antennas.[1]

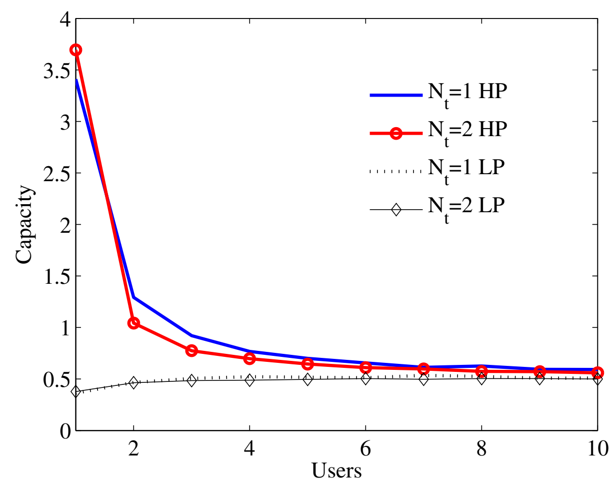

Fig. 2. Plot of Capacity v/s users for two different N_t for low power (LP, P=1) and high power (HP, P=100) for a Rayleigh fading MISO Multiple Access Channel.

IV. Delay Analysis

Having obtained the relationship between the capacity and the number of users, we can now evaluate the end-to-end delay for the MIMO MAC for a random channel. The scaled error exponent function is used as the service rate for queueing theoretic analysis. The variation of delay with users is plotted in Fig. [3] for a fixed $N_r = 3$ and varying N_t. Fig. [4] shows the delay profile for $N_t = 3$ and varying N_r. It is observed that increasing the number of transmit antennas for a fixed number of receive antennas only increases the delay, both under high

[1]When N_r is greater than 1, the variation of C with u exhibits a non-monotonic behavior similar to Figure 1 and is not shown here.

and low power. But, increasing the number of receive antennas for a fixed number of transmit antennas decreases the delay. Hence, for a Gaussian multiple access channel with no CSIT, increasing the number of transmit antennas does not improve performance. The reason for this increase in delay is the decrease in capacity as N_t increases for $u > 1$ (see Fig 2). To elaborate it further, the delay depends on the probability distribution of the number of users which in turn is inversely proportional to product of capacities(of 1 to u users). Hence the increase in delay with decrease in capacity.

Further, we observe from Fig. 3 and 4 that the change in capacity is more pronounced for varying N_r at a fixed N_t rather than for varying N_t for a fixed N_r. Using free probability theory, we have arrived at an analytical expression for the ergodic capacity under single user decoding(unpublished work). Even with this analytical expression, it is not straight forward to bring out the dependence of capacity on N_t or N_r. We plan on investigating this dependence analytically in our future work.

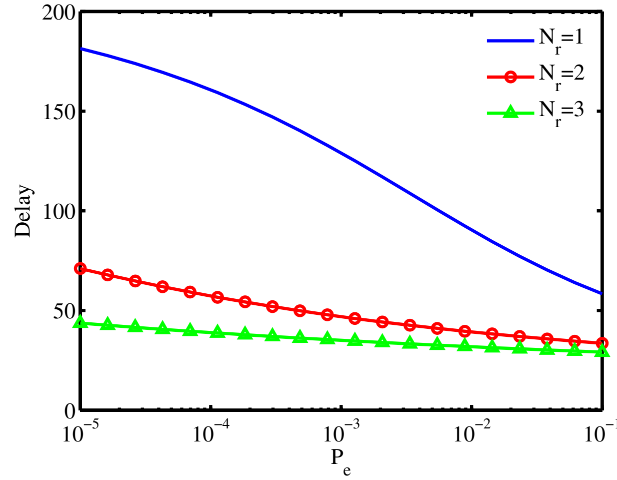

Fig. 4. Plot of Delay versus Probability of Error for $N_t = 3$, varying N_r for a Rayleigh Fading Multiple Access Channel.

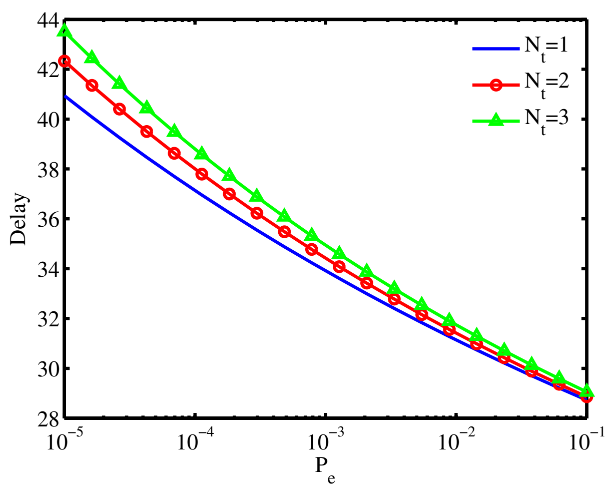

Fig. 3. Plot of Delay versus Probability of Error for $N_r = 3$, varying N_t for a Rayleigh Fading Multiple Access Channel.

V. Conclusions

In this paper, we derived results for multiantenna multiple access channels that are antithetical to the known results for the single user case and somewhat counter-intuitive for the multi user scenario too. Motivated by the necessity to unify concepts in physical layer and networking layer, we studied the variation of capacity versus the number of users in a multi user channel as a prelude to evaluating the end-to-end delay at the network layer. Having extended the error exponent to the MIMO MAC, we got insights on the variation of the capacity with the number of users having the transmit power and the number of transmit antennas as the parameters. Intuition would suggest that for single user decoding at a single antenna receiver, the sum-rate capacity will decrease monotonically as the number of users increases (due to increased interference). However, it turns out that under some constraints on the power and the number of transmit antennas, the capacity for deterministic channels can decrease and then increase with users. Further, for a Rayleigh fading MISO multiple access channel, numerical analysis indicates that the capacity decreases with increase in the number to transmit antennas. Further, for a MIMO Rayleigh fading channel, while increasing the number of receive antennas decreases the end-to-end delay, increasing the number of transmit antennas for a fixed number of receive antennas increases the delay!

References

[1] Anthony Ephremides. Some wireless networking problems with a theoretical conscience. In *Codes, Curves and Signals. Common threads in communications*, pages 201–212. Kluwer Academic publishers, 1998.

[2] Anthony Ephremides and Bruce Hajek. Information theory and communication networks: An unconsummated union. *IEEE Trans. on Information Theory*, 44(6):2416–2434, Oct. 1998.

[3] Robert G. Gallager. A perspective on multiaccess channels. *IEEE Trans. on Information Theory*, 31(2):124–142, Mar. 1985.

[4] I. E. Telatar and R. G. Gallager, "Combining Queueing Theory with Information Theory for Multiacess. " *IEEE Journal on Selected Areas in Communications*, Vol. 12, No. 6, pp. 963 - 969, Aug. 1995.

[5] Randall Berry and Edmund Yeh. Cross layer wireless resource allocation - fundamental performance limits. *IEEE Signal Processing Magazine*, July 2004.

[6] Sanjay Shakkottai, Theodore S. Rappaport, and Peter C. Karlsson. Cross-layer design for wireless networks. *IEEE Communications Magazine*, October 2003.

[7] L. Tong, V. Naware, and P. Venkitasubramanian. Signal processing in random access : A cross-layer perspective. *IEEE Signal Processing Magazine*, 21(5), September 2004.

[8] I. Emre Telatar, "Capacity of multi-antenna Gaussian channels," *European Trans. Telecomm.*, vol. 10, no. 6, pp. 585-595, 1999.

[9] R. Gallager, *Information Theory and Reliable Communication*. New York: Wiley, 1968.

[10] F. P. Kelly, *Reversibility and Stochastic Networks*. Wiley, New York, 1979.

[11] Dimitri P. Bertsekas and Robert Gallager. *Data Networks*. Prentice Hall, 1992.

[12] G. J. Foschini and M. J. Gans. On limits of wireless communications in a fading environment when using multiple antennas. *Wireless Personal Communication*, 1998.