

Tradeoff Between Source and Channel Coding for Erasure Channels

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Abstract—In this paper, we investigate the optimal tradeoff between source and channel coding for channels with bit or packet erasure. Upper and Lower bounds on the optimal channel coding rate are computed to achieve minimal end-to-end distortion. The bounds are calculated based on a combination of sphere packing, straight line and expurgated error exponents and also high rate vector quantization theory. By modeling a packet erasure channel in terms of an equivalent bit erasure channel, we obtain bounds on the packet size for a specified limit on the distortion.

Index terms—Joint source and channel coding, binary erasure channel, packet erasure, error exponent, high rate vector quantization.

I. INTRODUCTION

In [1] Shannon presented his celebrated result on the asymptotic optimality of separable source and channel coding. However, for finite block length systems, the importance and superior performance of joint source channel coders has been well recognized and is an area of active research (see for example [9]). Specific research thrusts have included investigating source coders that incorporate channel information in the design, channel coders that provide unequal error protection to various source bits, and iterative source-channel decoders.

An important issue in joint source channel coding is the tradeoff between source and channel coding rates. For a fixed source vector dimension and channel capacity, there is a tradeoff between the source and channel coding rates. A high rate channel code implies more bits for the source coder which results in a high quality representation at the source but has a higher probability of being received in error. Similarly, a low rate channel code results in fewer bits for the source coder; consequently, the representation is of lower quality at the source but there is a higher probability of being received without error at the receiver. This tradeoff has been quantified for binary symmetric channels (BSC) [2] and Gaussian channels [3].

This paper addresses the problem of optimal allocation of rate between a source encoder and a channel encoder for transmission over erasure channels. The system under investigation is a concatenation of a vector quantizer with a channel coder and the objective is to minimize the end-to-end distortion. Upper and lower bounds on the channel coding rate are constructed that minimizes the end-to-end distortion. The upper bound on channel coding rate is derived using the sphere packing and straight line exponents as a bound for

performance of the channel code. Similarly, the lower bound on rate is derived based on the expurgated error exponent for the erasure channel. The proposed bounds suggest that the optimal channel coding rate is substantially smaller than the channel capacity. Asymptotically, as the erasure probability $\epsilon \rightarrow 0$, the optimal channel coding rate equals 1. The resulting upper and lower bounds are then adapted to obtain the optimal coding rate for packet erasure channels.

The closed form approximations for the optimal coding rate are derived under the assumption of asymptotically small erasure probabilities. Also, a high rate quantization regime is considered and hence, distortion achieved asymptotically with large k equals the rate distortion bound. The proposed bounds are independent of the source distribution for sources with fixed dimensionality and finite support.

The rest of the paper is organized as follows. In Section 2, we define the system and present the notation and assumptions made in this paper. In Section 3 we evaluate the upper and lower bounds on the rate for erasure channels using the expurgated, sphere packing and straight line bounds. In Section 4 we present some numerical results and conclude in Section 5.

A model of the communication system under investigation is given in Fig. 1. Consider a random vector $X \in \mathfrak{R}^k$ that has a probability density function f over support set A , a closed bounded subset of \mathfrak{R}^k with nonempty interior. Let X be quantized by a vector quantizer $Q : \mathfrak{R}^k \rightarrow C$ where $C = [\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_M]$ is the codebook of the vector quantizer with $m = \log M$ bits per source symbol. All logarithms are to base 2. Consequently, the quantizer can be modeled as [4],

$$Q(\mathbf{x}) = \sum_{i=1}^M \mathbf{y}_i \mathbf{1}_{S_i}(\mathbf{x}) \quad (1)$$

where $\{S_i\}_{i=1}^M$ is a partition of \mathfrak{R}^k into disjoint regions, each of which is represented by \mathbf{y}_i and $\mathbf{1}_{S_i}(\cdot)$ is the indicator function which equals 1 if \mathbf{x} lies in the i^{th} cell of the partition. The average distortion using this quantizer is given by,

$$D_m(Q) = \sum_{i=1}^M \int_{S_i} \|\mathbf{x} - \mathbf{y}_i\|^p f(x) dx \quad (2)$$

$$= 2^{-pRr+O(1)}, \quad (3)$$

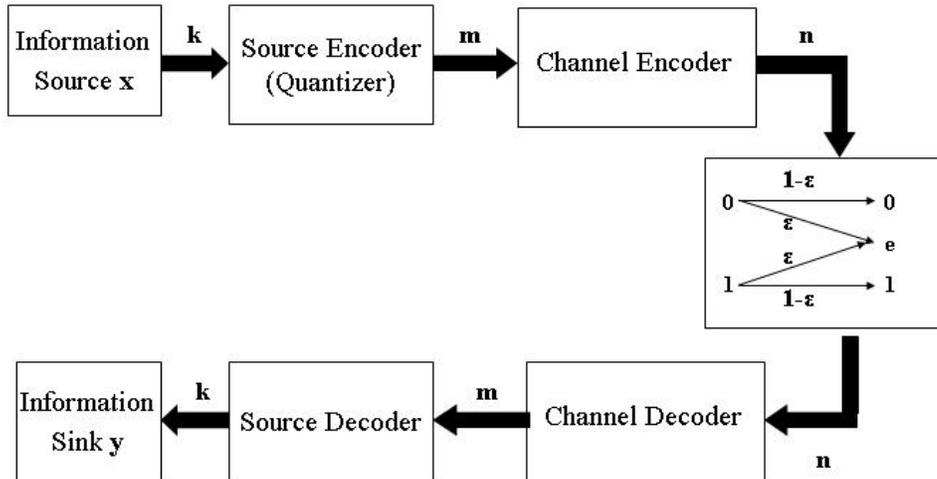


Fig. 1. System Block Diagram

where (3) follows from Zador's distortion formula [5] and p is the power of the distortion measure. Traditional quantization theory has worked on computing optimal quantizers that achieve the infimum of $D_m(Q)$. The quantizer design has been expanded to include the effect of channel errors; however, the problem is extremely challenging and little analytical results are known in such cases.

In our system, we consider that the m -bit source codewords are first randomly permuted using a mapping π and then passed to a channel encoder of rate $r = m/n$ before transmission over a binary erasure channel with erasure probability ϵ . For simplicity, we have included the index assignment π as part of the source encoder. The results are independent of the index assignment. The channel encoder generates a unique n -bit channel codeword for each of the m -bit source codewords. The added redundancy $n - m$ is used to protect the source code word from channel impairments. The transmission rate per source component is $R = n/k$, and the quantization rate is $R_s = m/k = Rr$. Following the notations used in [2], we denote $a_i = O(b_i)$ if $|a_i|/b_i \leq c$ for some $c > 0$ and i sufficiently large. We denote $a_i = \Omega(b_i)$ if $|a_i|/b_i \geq c$ for some $c > 0$ and sufficiently large i . Finally, $a_i = o(b_i)$ if $\lim_{i \rightarrow \infty} a_i/b_i = 0$.

II. BINARY ERASURE CHANNEL

We now consider obtaining bounds on the coding rate for a binary erasure channel (BEC). The end-to-end distortion for

the system in Fig. 1, is readily given by [2]

$$D_R(Q, \epsilon) = \sum_{i,j=1}^M q(j|i) \int_{S_i} \|\mathbf{x} - \mathbf{y}_i\|^p f(x) dx \quad (4)$$

where ϵ is the bit erasure probability and $q(j|i)$ is the conditional probability that the channel decoder decides in favor of the j^{th} channel codeword when the i^{th} codeword was transmitted.

A. Lower bound on channel coding rate

The lower bound on the channel coding rate is obtained by upper bounding the distortion at the decoder. Assuming small bit erasure probability and following [2], the total distortion may be upper bounded as,

$$D_R(Q, \epsilon) \leq D_m(Q) + O(1) \max_{1 \leq i \leq M} P_{e|i} \quad (5)$$

In (5), the total distortion is the sum of the distortion due to the vector quantizer ($D_m(Q)$), and the distortion due to the errors in transmission. The positive $O(1)$ term is due to the fact that f has support A , and y_j is contained in A for all j [2]. The problem of interest is posed as follows: "Given a binary erasure channel with R and source $X \in \mathbb{R}^k$, find the optimal rate r that minimizes the distortion $D_R(Q, \epsilon)$."

For an arbitrary binary discrete memoryless channel, Shannon's channel coding theorem guarantees that for channel

code rates r below capacity, the probability of error is upper bounded by [6]

$$\max_{1 \leq i \leq M} P_{e|i} \leq 2^{-nE_{ex}(r)+o(r)}, \quad (6)$$

where, $E_{ex}(r)$ is the expurgated error exponent and is an exponentially decreasing function of the rate. The dependence of $P_{e|i}$ on n indicates that the decoding error probability can be decreased by increasing the length of the channel codewords. The expurgated error exponent is given by [6],

$$E_{ex}(r) = \sup_{\rho \geq 1} \left[-\rho r + \max_{\mathbf{q}} E_x(\rho, \mathbf{q}) \right] \quad (7)$$

where,

$$E_x(\rho, \mathbf{q}) = -\rho \log \sum_{k=0}^{K-1} \sum_{i=0}^{K-1} q(k) q(i) \left[\sum_{j=0}^{J-1} \sqrt{P(j|k) P(j|i)} \right]^{1/\rho}. \quad (8)$$

Note that $\mathbf{q} = [q(0) q(1) \dots q(K-1)]$ represents the probability of the input channel alphabets and $P(j|i)$ is the probability of receiving output symbol j when input symbol i is transmitted. In (8), K and J represent, respectively, the cardinality of the input and output alphabets of the channel. For a binary erasure channel, $J = 3$ and $K = 2$. For the binary erasure channel, the transition probability matrix is given by

		Output, j		
		1	e	0
Input, i/k	1	$1 - \epsilon$	ϵ	0
	0	0	ϵ	$1 - \epsilon$

TABLE I

TRANSITION PROBABILITY MATRIX WITH ELEMENTS $P(j|i)$ FOR BEC.

For a symmetric channel, the \mathbf{q} that maximizes the error exponent is the uniform probability assignment [7]. Thus for the binary erasure channel, $\mathbf{q} = [q(0) q(1)] = [0.5 \ 0.5]$. Substituting the upper bound for the probability of error into (5), we obtain the end-to-end distortion as

$$D_R(Q, \epsilon, \pi) \leq 2^{-pRr+O(1)} + 2^{-kRE_{ex}(r)+o(r)} \quad (9)$$

Consider the case of large R : To ensure that neither of the two terms on the right hand side of (9) dominates the distortion upper bound, we choose the exponents of the two terms to be within $o(1)$ of each other [2], [8]. Hence, we set

$$E_{ex}(r) = \frac{p}{k} r_{ex} + o(1), \quad (10)$$

to obtain the channel coding rate that optimizes the end-to-end distortion at the decoder. This optimal rate is characterized by Theorem 1, which is similar to Theorem 1 in [2].

Theorem 1: The upper bound on the minimum p^{th} power distortion, averaged over all index assignments of a k -dimensional cascaded good vector-quantizer and channel encoder that transmits over a binary erasure channel with bit

erasure probability ϵ , is achieved with a channel code rate r_{ex} satisfying

$$r_{ex} = 1 - 2^{-c_\epsilon} \left(\frac{\log \log (1/\epsilon) + \log e + c_\epsilon}{\log (1/\epsilon)} \right) + O \left(\frac{\log \log (1/\epsilon)}{\log^2 (1/\epsilon)} \right) + o(1), \quad (11)$$

where, c_ϵ satisfies

$$\frac{p}{k} 2^{c_\epsilon} - \frac{(p/k) (\log \log (1/\epsilon) + \log e + c_\epsilon) - 2^{-c_\epsilon}}{\log (1/\epsilon)} - 1 = 0. \quad (12)$$

Proof:

For a BEC, evaluating (8), we obtain

$$\max_{\mathbf{q}} E_x(\rho, \mathbf{q}) = \rho \left[1 - \log \left(1 + \epsilon^{1/\rho} \right) \right] \quad (13)$$

and thus the expurgated error exponent becomes

$$E_{ex}(r) = \sup_{\rho \geq 1} \left\{ \rho \left[1 - r - \log \left(1 + \epsilon^{1/\rho} \right) \right] \right\} \quad (14)$$

The ρ which maximizes the error exponent and also satisfies (10) is given by

$$\rho = \frac{\log (1/\epsilon)}{\log \log (1/\epsilon) + c_\epsilon} \quad (15)$$

Substituting for ρ and c_ϵ into (10) we obtain the optimal rate (11) and hence the theorem is proved. \square

Note that the expression for the expurgated error joint source channel rate is similar to the BSC case [2] with the difference being the argument of the log term. Appendix 1 in [2] provides details on the derivation for ρ and c_ϵ . A further simplification in the expression for the rate can be obtained by neglecting the $O(1)$ and $o(r)$ terms and equating (14) to the exponent of the source coding distortion yielding

$$r_{ex} = \frac{\rho}{\frac{p}{k} + \rho} \left[1 - \log \left(1 + \epsilon^{1/\rho} \right) \right] \quad (16)$$

Numerical values of r_{ex} is given in Figure 2 and are explained in Section IV.

B. Upper bound on channel coding rate

Following the analysis in [2], the upper bound on the average distortion minimized over all channel code rates for large R and small bit error probability for the binary erasure channel can be obtained as

$$D_R(Q, \epsilon, \pi) \geq D_m(Q) (1 - P_e) + \Omega(1) \frac{1}{M} \sum_{k=1}^M P_{e|k} = 2^{-pRr+O(1)} (1 - P_e) + \Omega(1) P_e \quad (17)$$

where P_e is the probability of error occurring in the channel. A lower bound on this probability of error is given by,

$$P_e \geq 2^{-nE_{sl}(r)+o(n)} = 2^{-kRE_{sl}(r)+o(R)} \quad (18)$$

where E_{sl} is the straight line exponent. The straight line exponent $E_{sl}(r)$ is a linear function of r which is tangent

to the sphere packing exponent $E_{sp}(r)$ and also satisfies $E_{sl}(0) = E_{ex}(0)$. The sphere packing exponent [6] is given by,

$$E_{sp}(r) = \sup_{\rho \geq 0} \left[-\rho r + \max_{\mathbf{q}} E_o(\rho, \mathbf{q}) \right] \quad (19)$$

and,

$$\max_{\mathbf{q}} E_o(\rho, \mathbf{q}) = -\log \sum_{j=0}^J \left[\sum_{k=0}^K q(k) P(j|k)^{1/(1+\rho)} \right]^{1+\rho} \quad (20)$$

The straight line exponent can be written as,

$$E_{sl}(r_{sl}) = E_{ex}(0) + r_{sl} \frac{[E_{sp}(r') - E_{ex}(0)]}{r'} \quad (21)$$

where, r' is the rate at which the straight line exponent meets the sphere packing exponent tangentially. The straight line exponent has also been characterized in [10] for a binary erasure channel. Now, the end-to-end distortion is thus bounded as

$$D_R(Q, \epsilon, \pi) \geq 2^{-pRr+O(1)} + 2^{-kRE_{sl}(r)+o(R)} \quad (22)$$

The channel coding rate that minimizes this bound is now characterized in Theorem 2.

Theorem 2: An upper bound on the channel code rate r that minimizes the p^{th} power distortion averaged over all random index assignments of a k -dimensional cascaded good vector quantizer for a binary erasure channel with small effective bit erasure probability ϵ and large R is given by

$$r_{sl} = \frac{E_{ex}(0)}{\frac{p}{k} - \frac{E_{sp}(r') - E_{ex}(0)}{r'}} \quad (23)$$

Proof: As in the earlier case, for large R , to prevent either of the terms in the distortion bound (22) from dominating the other, we set the straight line exponent to be linearly proportional to the exponent term of the noiseless-optimal distortion within $o(1)$ of each other. Thus,

$$E_{sl}(r) = \frac{p}{k} r_{sl} + o(1) \quad (24)$$

Substituting (21) in (24), the theorem is proved \square .

Note that to completely characterize r_{sl} we need to explicitly evaluate the sphere packing exponent $E_{sp}(r)$. It is easily seen that a uniform probability assignment for the input states to the channel $q(\cdot)$ maximizes $E_o(\rho, \mathbf{q})$ and thus $E_{sp}(r)$ can be evaluated as,

$$E_{sp}(r) = \sup_{\rho \geq 0} \{ \rho(1-r) - \log[(1-\epsilon) + \epsilon 2^\rho] \} \quad (25)$$

Note that (25) is a concave function of ρ and hence the supremum can be replaced by the max operator. The ρ which maximizes (25) satisfies

$$r = \frac{(1-\epsilon)}{(1-\epsilon) + 2^\rho \epsilon} \quad (26)$$

We can use this relation between the rate and ρ to express the sphere packing exponent in terms of the channel encoding

rate for a given erasure channel as,

$$E_{sp}(r) = r \log r + (1-r) \log(1-r) - r \log \left(\frac{1-\epsilon}{\epsilon} \right) - \log \epsilon \quad (27)$$

At r' , the slope of the sphere packing exponent equals the slope of the straight line exponent. Thus,

$$\frac{\partial E_{sp}}{\partial r} \Big|_{r=r'} = \frac{E_{sp}(r') - E_{ex}(0)}{r'} \quad (28)$$

Differentiating (27) and substituting in (28), we get

$$r' = 1 - 2^{E_{ex}(0) - \log(1/\epsilon)} \quad (29)$$

It turns out that $E_{sp}(r')$ is nearly 0 for small values of ϵ .

III. NUMERICAL RESULTS

The bounds derived above for the erasure channel can be easily extended to the case of packet erasures. We use a simplified model for the packet erasure channel and assume that a packet erasure occurs if any of the bits within the packet suffers an erasure. Although this assumption simplifies the packet erasure channel model, it is useful in obtaining closed form bounds on the coding rate over such channels. For a packet of size P bits, the probability of a packet erasure δ is given by $\delta = 1 - (1-\epsilon)^P$, where as before ϵ denotes the probability of bit erasure. The error exponent for a binary erasure channel with erasure probability ϵ and a 2^P -ary erasure channel with erasure probability δ is the same. Hence, given the packet erasure probability δ , we consider an equivalent binary erasure channel with bit erasure probability $\epsilon = 1 - (1-\delta)^{1/P}$ and find the bounds on the rate and distortion for the corresponding BEC.

The plot of the upper and lower bound on channel coding rate as a function of the erasure channel probability for $k=4$ and squared distortion measure is given in Fig. 2. The bounds for various packet sizes $P = 1, 10$ and 100 are shown in Fig. 2. It is observed that for a given packet size, as the erasure probability increases, the channel coding rate decreases indicating that more bits need to be invested on channel coding to combat a hostile channel. Further, for a given packet erasure probability, as the packet size increases, the channel coding rate increases implying that more bits can be allocated for source coding with larger packet size. Fig. 3 offers a different perspective on the results. From the bounds on the channel coding rate, we can get the bounds on the distortion due to channel coding. By virtue of our optimal joint source-channel coding criterion, the total distortion will be twice the distortion due to channel coding. Hence, given an end-to-end limit on the distortion, we can get the minimum packet length to be chosen from Fig. 3. The squared distortion metric with $k = 4$, $R = 10$ and packet erasure probability $\delta = 10^{-3}$ was chosen. The $o(r)$ and $o(R)$ terms were neglected in the terms for distortion due to noisy channel decoding in (9) and (22). The asymptotic nature of the curve indicates that large packet size is not required for packet erasure channels with small erasure probabilities.

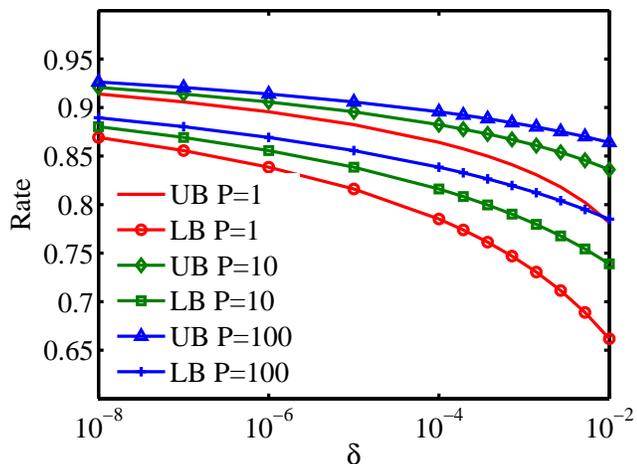


Fig. 2. The upper (UB) and lower bounds (LB) on the optimal channel coding rate are plotted for various values of packet size P .

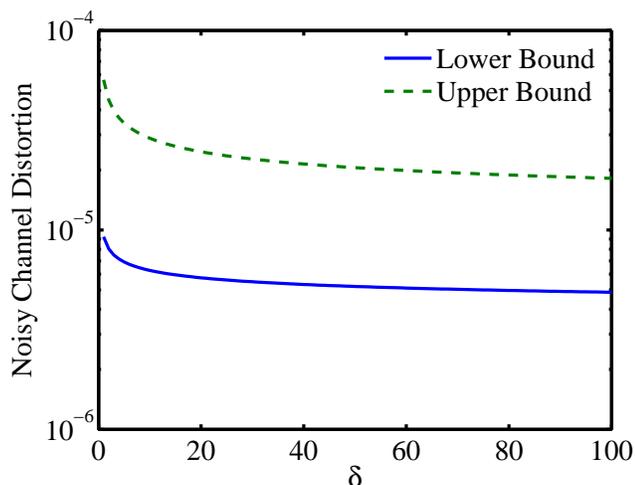


Fig. 3. The Distortion for various packet lengths for a packet erasure channel with packet erasure probability $\delta = 10^{-3}$ and $R = 10$.

IV. CONCLUSION

The results presented in this paper provide a mechanism for optimal concatenation of source and channel coders. Analytic results are provided for lower and upper bounds for a binary erasure channel and for packet erasure channels. The results on packet erasure channel enable us to obtain the bounds on the packet size for a specified bound on the distortion and given packet erasure probability. Alternately, for a given packet erasure probability, we can find bounds on the channel encoding rate for various packet lengths. By studying the optimal rate allocation for a bit and packet erasure channel, one can apply these results for transmission in a wide range of scenarios, including wireline channels with congestion. In future work, these bounds should be expanded to include transmission over more sophisticated channel models.

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